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in
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Determining the Seifert genus via the Heegaard Floer tangle invariant HFT

Author: *Benedikt Aubeck*
Matriculation number: 1855440
Supervisors: *Dr. Lukas Lewark* & *Dr. Claudius Zibrowius*
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Table of Contents

Notation

$\operatorname{im} f$	image of the map f
$A \setminus B$	relative complement of B in A
I	unit interval
S^n	n-dimensional unit sphere
B^n	n-dimensional unit ball
$\sqcup$, $\bigsqcup$	disjoint union
∂M	border of the manifold M
$\lvert A \rvert$	cardinality of the set A
$\mathbb{N}$	the natural numbers (containing zero)
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{F}_2$	the field with two elements
$\mathbb{Q}\mathrm{P}^1$	the projective line over $\mathbb{Q}$
$\mathrm{SL}_2(\mathbb{Z})$	the special linear group of degree 2 over $\mathbb{Z}$
$\mathrm{PSL}_2(\mathbb{Z})$	the projective special linear group of degree 2 over $\mathbb{Z}$
$\lvert a \rvert$	absolute value of $a \in \mathbb{R}$
$\oplus$, $\otimes$	direct sum and tensor product over $\mathbb{F}_2$
U_n	n-component Unlink
$Q_{p/q}$	rational tangle with slope $p/q \in \mathbb{Q}\mathrm{P}^1$
$X \cup Y$	union of two Conway tangles
S^2_4	four-punctured sphere
$\mathrm{Mod}(S^2_4)$	mapping class group of S^2_4
$\mathfrak{r}(p/q)$	rational curve of slope p/q
$\mathfrak{s}_n(p/q; x, y)$	special curve of slope p/q through the punctures x and y
$\mathrm{HF}(\gamma, \gamma')$	Lagrangian Floer homology of the curves γ and γ'
For an (oriented) link L:	
$\mu(L)$	number of components/multiplicity of L
$g(L)$	genus of L
$\Delta_L(t)$	Alexander polynomial of L
$\widehat{\mathrm{HFK}}(L)$	knot Floer homology of L
$\widehat{\mathrm{HFK}}_*(L)$	knot Floer homology of L with collapsed Maslov grading
For an (oriented) Conway tangle T:	
$\mathrm{m}(T)$, $\mathrm{mr}(T)$	mirror and reversed mirror of T
$N(T), D(T)$	numerator and denominator of T
$T(p/q)$	p/q-closure of T
$\mathrm{HFT}(T)$	bigraded multicurve invariant associated to T

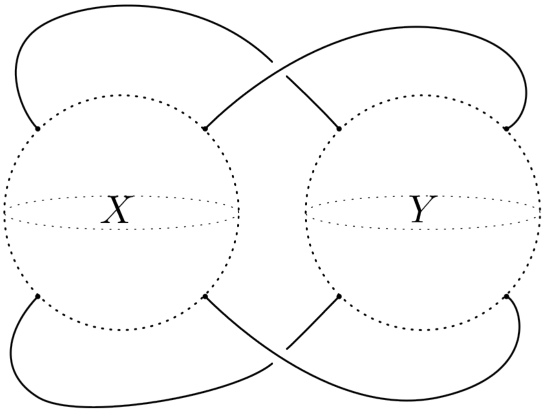


Figure 1: The link $X \cup Y$ constructed from tangles X and Y

1 Introduction

The prerequisite of any classification is the ability to distinguish. In mathematics, we like to distinguish objects by calculable invariants. Hence also in knot theory, where we use *link invariants* to distinguish knots and links and to classify them in tables. Various of these link invariants have been developed over the last century, some more topologic, some more algebraic in nature. The simplest example would be the number of components of a link. Other classical examples are the *crossing number*, the *determinant* or *colourability* of a link.

In the 1960s John Conway had another idea how to classify links. Using an embedded 2-sphere he decomposed links into two separate components, the so-called *tangles* (Section 2.1). If we restrict ourselves to 2-spheres intersecting a link L exactly four times, we can think about a tangle decomposition as union

$$L = X \cup Y$$

depicted in figure 1, where X and Y are called *Conway tangles*. Conway showed that if these Conway tangles satisfy a certain triviality condition, we can classify them as *rational tangles* Q_s for $s \in \mathbb{Q}\mathrm{P}^1$ (Section 2.3). This algebraic classification gives an algorithm (Conway's algorithm) which associates a tangle to a continued fraction. If a tangle decomposition splits a link into two rational tangles, we call this link a *rational link*. Such an oriented rational link can be described with a particular continued fraction called the *cyclical even continued fraction* which contains certain informations about the link (Section 3.3). Furthermore, Horst Schubert showed that local classifications of rational links can be used to classify the whole link (section 2.4).

At this point, we learn about the *Seifert genus*, an important link invariant which was introduced by Herbert Seifert (Sections 3.1 and 3.2). Per definition the genus of an oriented link is the minimal genus of any compact, oriented and connected surface that has the given link as oriented border. However, there are more convenient ways to

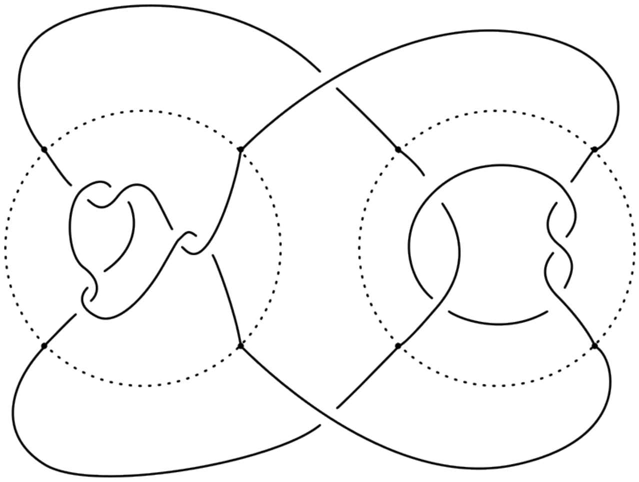


Figure 2: The rational closure $T_{2,-3}(-4/3) = Q_{4/3} \cup T_{2,-3}$

compute the Seifert genus, for example the following formula for which we will give an alternative proof using recent methods in section 5.3.

Proposition 1.1 (Proposition 3.17)**.** *Let L be an oriented rational link with cyclical even continued fraction $[a_1, \ldots, a_n]$. Then it holds*

$$g(L) = \begin{cases} 1/2\, n, & \textit{if } n \textit{ is even} \\ 1/2\,(n-1), & \textit{if } n \textit{ is odd} \end{cases}$$

if $a_1 \neq 0$ or $n \leq 1$. Otherwise

$$g(L) = \left.\begin{cases} 1/2\, n, & \textit{if } n \textit{ is even} \\ 1/2\,(n-1), & \textit{if } n \textit{ is odd} \end{cases}\right\} - 1$$

holds.

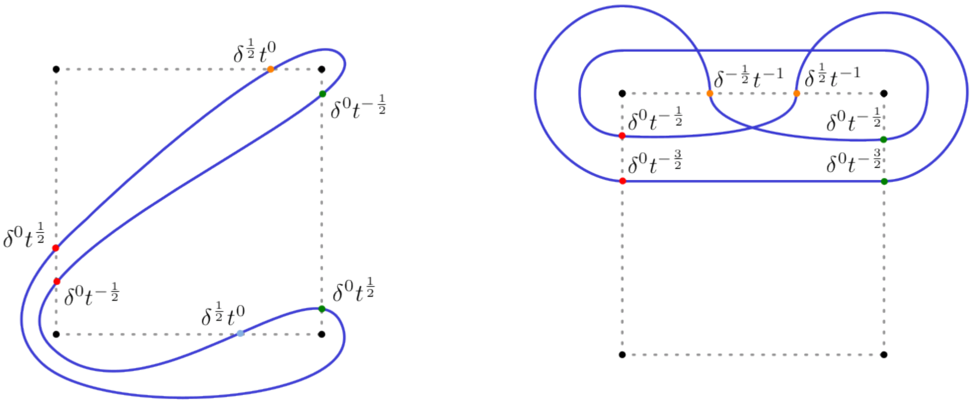
In this work we are particularly interested in *rational closures* of tangles, i.e. given an oriented Conway tangle T we look at the links

$$T(s) := Q_{-s} \cup T$$

for suitable $s \in \mathbb{Q}\mathrm{P}^1$. For the (2, -3)-pretzel tangle $T_{2,-3}$ we can see the exemplary rational closure $T_{2,-3}(-4/3)$ in figure 2. With this notion, rational links are but rational closures of rational tangles. Using the above formula we're able to compare the genera of rational closures of two given oriented rational tangles (Section 3.3). More precise we prove the following two lemmas.

Lemma 1.2 (Lemma 3.22)**.** *Let Q_0 and $Q_{1/2}$ be oriented cyclically. Let $p/q \in \mathbb{Q}\mathrm{P}^1$ with p **even** and q **odd**. Then it holds*

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) \quad \textit{for} \quad p/q \in \{0\} \cup (1/3,\, 1)$$



(a) The rational curve $\mathrm{HFT}(Q_{1/2})$

(b) The special curve $\mathfrak{s}_1(0; 1, 4)$

Figure 3: Bigraded immersed curves

and

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) - 1 \quad \textit{otherwise.}$$

Lemma 1.3 (Lemma 3.23)**.** *Let Q_0 and $Q_{1/2}$ be oriented cyclically. Let $p/q \in \mathbb{Q}\mathrm{P}^1$ with p* ***odd*** *and q* ***even****. Then it holds*

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) + 1 \quad \textit{for} \quad p/q \in (1/3,\, 1) \setminus \{1/2\}$$

and

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) \quad \textit{otherwise.}$$

In the early 2000s Peter Ozsváth and Zoltán Szabó developed mighty new knot and link invariants called *Heegaard Floer homologies*. Of particular use was a invariant called the *knot Floer homology* $\widehat{\mathrm{HFK}}$ (Section 4.1), which was independently discovered by Ozsváth-Szabó and Jacob Rasmussen. For us it takes the form of a bigraded vector space with respect to a δ-grading and an Alexander grading. Further we list results from Yi Ni, which enable us to determine the genus and fibredness of an oriented link using its knot Floer homology. One might ask now, whether we can find a connection between the knot Floer homology of a link and a tangle decomposition of it.

A few years ago Claudius Zibrowius introduced such a connection using a ”local” knot Floer homology called *tangle Floer homology* together with a gluing theorem to recover knot Floer homology. In the case of a Conway tangle T, this tangle Floer homology can be described using a collection of decorated relatively bigraded immersed curves $\mathrm{HFT}(T)$ on a parametrized four-punctured sphere S^2_4 (Section 4.2). This computable tangle invariant HFT carries the information of a relative δ-grading and Alexander grading on its intersections with a certain auxiliary parametrization of S^2_4. In figure 3 we see two such bigraded curves on a parametrized (dotted lines) four-punctured sphere (which is depicted as a four-punctured plane plus a point at infinity). With this relative bigrading and the geometry of the immersed curves we can distinguish tangles and determine

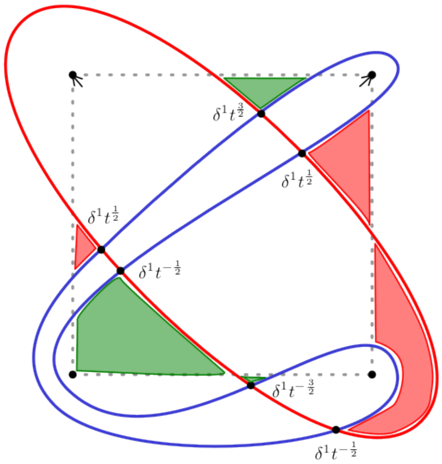


Figure 4: Pairing of $\mathrm{HFT}(Q_{-1})$ and $\mathrm{HFT}(Q_{1/2})$

certain properties, e.g. whether the tangle is rational. Furthermore, this local version of knot Floer homology satisfies a gluing property (Section 4.4) involving the Lagrangian Floer homology HF of the multicurves invariants (Section 4.3). For this Lagrangian Floer homology we unfortunately had to assume a reasonable conjecture to be true (Conjecture 4.47). The gluing can be interpreted geometrically by intersecting the curves and deriving a bigrading on the intersection points (Figure 4). To achieve more control over this gluing process we introduce a way to symmetrize the Alexander grading. In particular we show the following two results.

Theorem 1.4 (Theorem 4.68)**.** *Let Q be a oriented rational tangle. Then* $\mathrm{HFT}(Q)$ *has a symmetric Alexander grading.*

Corollary 1.5 (Proposition 4.70 and lemma 4.57)**.** *Let $L = Q_s \cup Q_t$ be an oriented rational link. If we fix the symmetric Alexander grading on* $\mathrm{HFT}(\mathrm{mr}(Q_s))$ *and* $\mathrm{HFT}(Q_t)$ *then the relative isomorphism from the gluing theorem*

$$\widehat{\mathrm{HFK}}(L) \otimes V^{2-\mu(L)} \;\cong\; \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(Q_s)), \mathrm{HFT}(Q_t))$$

is absolute with respect to the Alexander grading.

In the next section, we examine the knot Floer homology of rational links using the multicurve invariant. In particular, we deduce a general formula (Section 5.1) for the computation and prove already existing formulas using new methods (Sections 5.2 and 5.3). Along the way we give an alternative proof for a formula found by Hartley and Minkus for the Alexander polynomial $\Delta_L(t)$ of a rational link L.

Theorem 1.6 (Hartley; Minkus. Theorem 5.12)**.** *Let $L = Q_0 \cup Q_{p/q}$ be an oriented rational link as in remark 5.9. Then*

$$\Delta_L(t) \doteq \sum_{k=0}^{p-1} (-1)^k t^{\sum_{i=0}^{k} \varepsilon_i}$$

where $\varepsilon_i \coloneqq (-1)^{\lfloor iq/p \rfloor}$.

Furthermore, we prove the convenient genus formula for rational links from above and the following statement about the number of generators in the highest Alexander degree.

Proposition 1.7 (Proposition 5.20)**.** *Let L be a rational link with cyclical even continued fraction* $[a_1, \ldots, a_n]$*. Then for*

$$A_{\max} \coloneqq \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L, A) \neq 0\}$$

holds that

$$\dim \widehat{\mathrm{HFK}}_*(L, A_{\max}) = \begin{cases} 2, & \text{if } n = 1 \wedge a_1 = 0, \\ \prod_{i=3}^{n} |a_i|/2, & \text{if } n > 1 \wedge a_1 = 0, \\ \prod_{i=1}^{n} |a_i|/2, & \text{if } n = 0 \vee a_1 \neq 0. \end{cases}$$

In the last section 6, we want to examine rational closures of the (2, -3)-pretzel tangle $T_{2,-3}$. We therefore look at its multicurve invariant $\mathrm{HFT}(T_{2,-3})$ (Section 6.1) and reduce the complexity to computations concerning only rational links (Section 6.2). To achieve this, we have to take a look at pairings with special curves like the one in figure 3b. We start our conclusion (Section 6.3) with the results

Proposition 1.8 (Proposition 6.8)**.** *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with p even and q odd. Then*

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \end{array}$$

holds.

Proposition 1.9 (Proposition 6.9)**.** *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with p odd and q even. Then*

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2} \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2} \end{array}$$

holds.

which lead to our main theorem and its corollaries.

Theorem 1.10 (Theorem 6.10)**.** *Let* $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with* $p + q \equiv 1 \pmod 2$ *and let*

$$c(p/q) = \begin{cases} 1, & \text{if } p \text{ is odd, } q \text{ is even} \\ 2, & \text{if } p \text{ is even, } q \text{ is odd.} \end{cases}$$

Then we have

$$g\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \max\left\{ \begin{array}{l} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + c(p/q) \end{array} \right\}.$$

Corollary 1.11 (Corollary 6.11)**.** *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with* $p + q \equiv 1 \pmod 2$ *and let*

$$c(p/q) = \begin{cases} 1, & \text{if } p \text{ is odd, } q \text{ is even} \\ 2, & \text{if } p \text{ is even, } q \text{ is odd.} \end{cases}$$

Then we have

$$g\left(T_{2,-3}(p/q)\right) = g\left(Q_0(p/q)\right) + c(p/q).$$

Corollary 1.12 (Corollary 6.14)**.** *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with* $p + q \equiv 1 \pmod 2$*. Then*

$$T_{2,-3}(p/q) \text{ is fibred} \iff Q_0(p/q) \text{ is fibred.}$$

We are using the basic definitions for knots and links from [Cro+04]. In particular, we assume that all links lie in S^3 if not otherwise mentioned. All vector spaces are supposed to be over $\mathbb{F}_2$. All embeddings shall be locally flat, in particular all links are tame.

At the end of this introduction, I would like to express my sincere **thanks** to my supervisors Claudius Zibrowius and Lukas Lewark, who willingly put up with my questions and supported me far beyond what anyone could have wished for!

2 Tangles

The notion of tangle was introduced by John Conway in the 1960s to simplify the process of classifying (also called enumerating) links. He succeeded by means of rational tangles respectively links and was able to compute the existing link tables significantly faster than was possible before ('in an afternoon' as he himself said in [Con70]).

2.1 Conway tangles

Idea 2.1. Think about a link L in S^3 and an embedded 2-sphere $S \subset S^3$, which intersects L transversely. We think about the two components of $S^3 \setminus S$ as tangles, because they are a (possible empty) collection of embedded arcs and loops.

If we could now classify this tangles (in a sense we classify L *locally*) we might be able to classify L as a whole (i.e. *globally*). This idea will lead to the concept of rational tangles, the simplest type of tangles, and their classification.

Later on in this work, we will use the very same idea to determine the genus and fibredness of links: We will use invariants of the two tangles and put these together in an elegant way to get an invariant of the link.

Definition 2.2 (Tangle)**.** Let B be a closed 3-ball and $n, m \in \mathbb{N}$. An n**-tangle** T is a proper embedding

$$T : (\bigsqcup_n I \sqcup \bigsqcup_m S^1, \bigsqcup_n \partial I) \hookrightarrow (B, \partial B).$$

We call $T(\bigsqcup_n I)$ the **open components** and $T(\bigsqcup_m S^1)$ the **closed components** of T. The $2n$ points $\partial T \coloneqq T(\bigsqcup_n \partial I)$ are called the **tangle ends**. For this reason we also speak of $2n$-**ended tangle** for an n-tangle. Given a tangle T we denote its encircling closed 3-ball by B_T. We call ∂B_T the **boundary of the tangle** T.

Hence, an n-tangle can be thought of as a closed 3-ball containing n properly embedded arcs, woven together with a (possibly empty) number of properly embedded loops, such that the arcs and loops are disjoint.

Remark 2.3. The introductory idea to think about an n-tangle is the following: An n-tangle is a pair (B, T) of subspaces of S^3 such that there exists a link L and a closed 3-ball $M \subset S^3$ whose boundary intersects L transversely in $2n$ points such that the pair $(M, M \cap L)$ is homeomorphic to (B, T).

In fact, given an n-tangle T the pair $(B, \operatorname{im} T)$ satisfies this. We see this by embedding B into S^3 and closing the tangle ends by proper arcs in the complement of the interior of B in an arbitrary manner. On the other hand a pair (B, T) as above can be thought of to be the associated pair $(B, \operatorname{im} T)$ of an n-tangle T. As B intersects the links transversely, we can clearly find such an n-tangle.

Remark 2.4. Let S be a 2-sphere embedded in S^3. Then $S^3 \backslash S$ consists of two connected components bounded by S [Cro+04, p. 37, Fact 2.4.5]. In fact, as the embedding is locally flat Schoenflies' theorem for three dimensions shows that S bounds a 3-ball on both sides [Cro+04, p. 38, Fact 2.4.6].

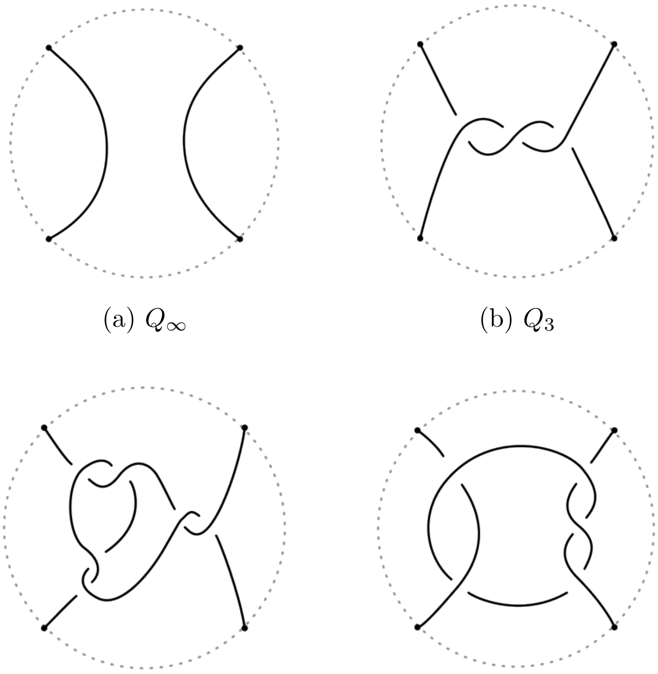
Any 2-sphere which intersects a link L transversely in S^3 therefore gives a **tangle decomposition** of L, i.e. a decomposition into two tangles.

Definition 2.5 (Trivial tangle)**.** A n-tangle T is called **trivial** if $(B_T, \operatorname{im} T)$ is homeomorphic as a pair to a cylinder $(D^2 \times I, \{x_1, \ldots, x_n\} \times I)$ for $x_1, \ldots, x_n \in D^2$, $x_i \neq x_j$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$.

Hence trivial tangles necessarily have no closed components. Even if our definition of tangle is the natural way to introduce this notion, it has certain problems:

Example 2.6. Some examples for 2-tangles can be seen in figure 5. The notation becomes clear later on. Mind that Q_∞, Q_3 and $Q_{4/3}$ are actually trivial. However, the $(2, -3)$-pretzel tangle $T_{2,-3}$ is *not* trivial as we will see later in remark 2.44.

Problem 2.7 (The loose border)**.** Even though our definition seems organic, it has an inherent problem. If we consider tangles like in the case of triviality up to homeomorphism, all trivial tangles are considered to be equal. However, as seen in figure 6, replacing a trivial tangle (in blue) in a tangle decomposition with another one can easily change the link type. This behaviour is undesirable, so we want to attain more control over the tangles to be able to distinguish trivial tangles. This could be done setting up the tangle border in a certain way and only considering homeomorphisms with fixed border. Yet, we will use an embedded auxiliary circle and a labelling of the tangle ends. For the sake of simplicity, we restrict ourselves to 2-tangles in the following definition.



(a) Q_∞ (b) Q_3 (c) $Q_{4/3}$ (d) $T_{2,-3}$

Figure 5: Examples of 2-tangles

Definition 2.8 (Conway tangle)**.** Let B be a closed 3-ball and $m \in \mathbb{N}$. A **Conway tangle** T is a proper embedding

$$T: \left(I \sqcup I \sqcup \bigsqcup_m S, \partial I \sqcup \partial I\right) \hookrightarrow (B, S \subset \partial B),$$

such that the endpoints of the two intervals lie on a fixed oriented circle S on the boundary of B, together with a choice of a distinguished tangle end $* \in \partial T := T(\partial I \sqcup \partial I)$. Starting at this distinguished (= first) tangle end and following the orientation of the fixed circle S, we number the tangle ends by 1, 2, 3, 4 and label the arcs $S \setminus \operatorname{im}(T)$ by a, b, c and d. We call a choice of a single arc a **site** of the tangle T.

We consider Conway tangles up to ambient isotopy which fixes the distinguished tangle end and the orientation of S (and thus preserves the labelling of the tangle ends). An **orientation of a Conway tangle** is a choice of orientation of the two arcs and the loops. Given an oriented Conway tangle the tangle ends point either **inwards** or **outwards** depending on whether the orientation of the open component points into or out of the 3-ball. We call two Conway tangles **equally oriented**, if tangle ends with the same label point in the same direction.

Note that the orientation of S enables us to distinguish between the two components of $\partial B \setminus S$. The **back component** of $\partial B \setminus S$ is the one whose boundary orientation agrees with the orientation of S, using the right-hand rule and a normal vector field pointing

(a) Unknot (b) Left-handed trefoil knot

Figure 6: Loose border problem

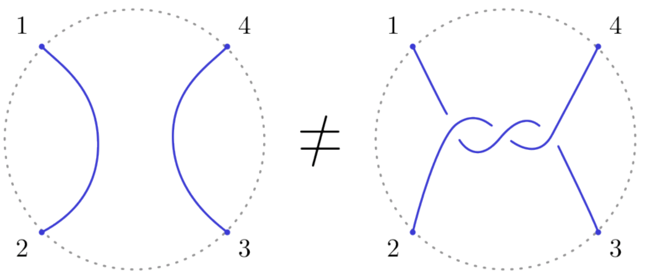


Figure 7: Different Conway tangles

into B. We call the other one the **front component**.

Remark 2.9. It is clear by embedding an axillary circle, that every 2-tangle is homeomorphic to a Conway tangle (in fact to multiple). For example we could interpret the tangles from in figure 5 as Conway tangles by labelling the four tangle end by $1, 2, 3, 4$. On the other hand, every Conway tangle clearly satisfies the definition of a 2-tangle.

Remark 2.10 (Loose border no more)**.** Examine in figure 7 that after setting suitable tangle end labels (and a fitting S) we can now simply distinguish the two trivial tangles from problem 2.7 by comparing to which tangle end the distinguished tangle end 1 connects (2 vs. 3 in this case). This criterion works, because we consider by definition 2.8 only ambient isotopies which fix the distinguished tangle end preserve the labelling of the tangle ends.

The name "Conway tangle" is reasonable when we think about Conway spheres.

Definition 2.11 (Conway sphere)**.** Let L be a link. A **Conway sphere** $S \subset S^3$ is a 2-sphere such that S intersects L transversely in four points.

Conway spheres therefore give rise to two 2-tangles. On the other hand, a 2-tangle can always be thought of as resulting from a Conway sphere.

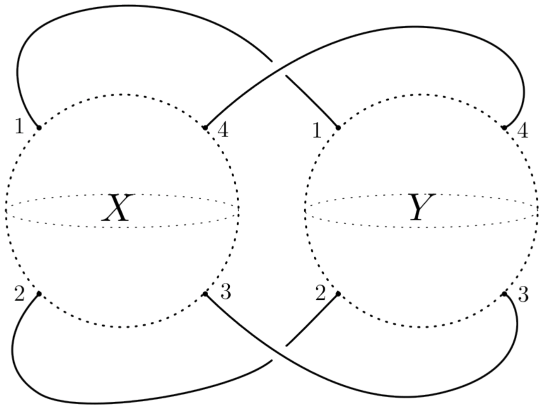


Figure 8: The union $X \cup Y$

Definition 2.12 (Union of Conway tangles)**.** Let X and Y be Conway tangles. The **union $X \cup Y$ of X and Y** is the link we obtain by connecting X and Y as in figure 8. We can also think about the union as X and Y being glued together along $\partial B_X \cong \partial B_Y$ such that the tangle ends are glued together according to their labels. One checks by rotations that $X \cup Y = Y \cup X$.

Remark 2.13. With this notion we can always think of a 2-tangle decomposition as the union of two Conway tangles parametrized in a suitable way.

Later we will sometimes mention the following notion.

Definition 2.14 (Split tangle)**.** A tangle T is called **split** if $B_T \setminus \operatorname{im} T$ is compressible.

Remark 2.15. The notion of a split tangle can be related to the notion of a split link. Let T_1, T_2 be Conway tangles without closed components and $L = T_1 \cup T_2$ (and think about L as glued together along a single 2-sphere $R := \partial T_1 \cong \partial T_2$). If L is split, we find a 2-sphere S separating the two components of L and which is transverse to R. We can now argue as in the proof of [Zib20, Lemma 6.3] to show that T_1 or T_2 is split.
The converse statement is false, as we will see later in corollary 2.42. Even if both tangles are split, the union might be non-split.

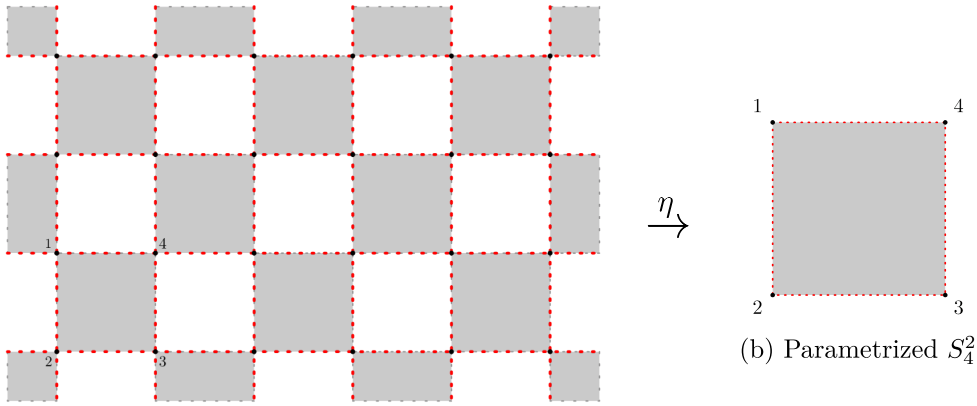
2.2 A covering space for the four-punctured sphere

Given a Conway tangle T removing the four tangle ends from its boundary gives us a four-punctured sphere. We get a covering space for this four-punctured sphere

$$S^2_4(T) := \partial B_T \setminus \partial T$$

by considering the planar covering that factors through the toroidal two-fold cover

$$\mathbb{R}^2 \to T^2 \to S^2 \cong \partial B_T$$



(a) Parametrized $\mathbb{R}^2 \setminus \mathbb{Z}^2$

(b) Parametrized S^2_4

Figure 9: The covering map η

which is defined as the composition of the universal covering of the torus $T^2 = \mathbb{R}^2/(2\mathbb{Z})^2$ with the double branched covering $T^2 \to S^2$, branched at the four puncture points of S^2. If we restrict this map to the preimage of S^2_4, we obtain the covering

$$\eta : \mathbb{R}^2 \setminus \mathbb{Z}^2 \to S^2_4.$$

From our tangle T we not only get such a four punctured sphere: The embedded fixed circle $S \subset \partial B$ gives rise to the four embedded arcs a, b, c, d connecting the punctures (which inherit the respective tangle end label). These arcs are called the **auxiliary parametrization** of S^2_4.

Under our covering map η this auxiliary parametrization lifts to a parametrization of our total space $\mathbb{R}^2 \setminus \mathbb{Z}^2$ which looks like a wobbly grid in $\mathbb{R}^2$ going through the points $\mathbb{Z}^2$. However, we can always ambient isotope T (fixing the distinguished tangle end) until the preimage looks like the standard unit grid. Hence we will always assume that S has this form, illustrated in figure 9, where the parametrization of S^2_4 has been lifted to $\mathbb{R}^2 \setminus \mathbb{Z}^2$ and the front face and its preimage under η are shaded grey. Mind that the four-punctured sphere is shown on the right as the plane with a point at infinity.

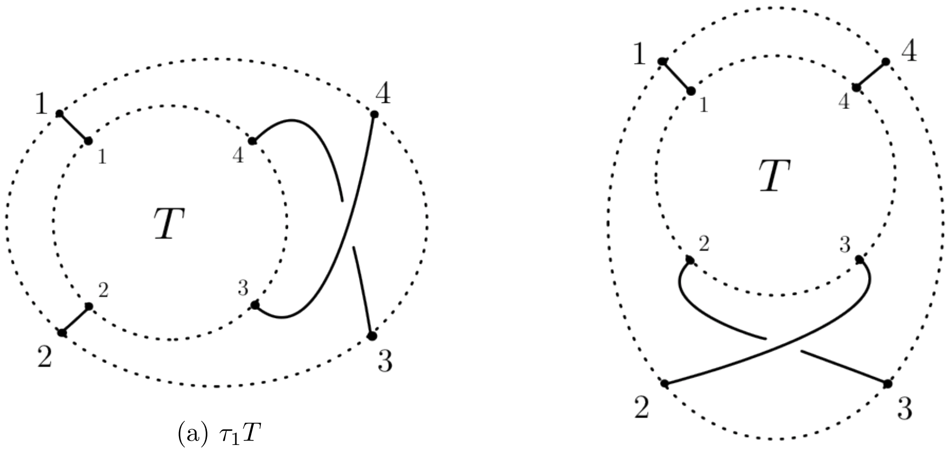
It is interesting to look at the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2 \setminus \mathbb{Z}^2$, whose elements can be thought of as orientation preserving homeomorphism of $\mathbb{R}^2 \setminus \mathbb{Z}^2$. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two shearings

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which induce two half-twists on S^2_4. Therefore the whole action induces an action on S^2_4 [Zib19, Observation 3.2]. To specify this we introduce the following notion.

The **mapping class group** of S^2_4 is defined as

$$\mathrm{Mod}(S^2_4) := \pi_0(\mathrm{Homeo}^+(S^2_4))$$



(a) $\tau_1 T$

(b) $\tau_2 T$

Figure 10: Generating half-twists

meaning the isotopy classes of orientation preserving automorphisms of S^2_4. This set admits a group structure given by composition and acts on the four-punctured sphere. In fact one can compute:

Theorem 2.16 ([Cro+04, p.199, Section 8.5])**.** *The mapping class group* $\mathrm{Mod}(S^2_4)$ *of* S^2_4 *is* $\mathrm{PSL}_2(\mathbb{Z})$.

Fact 2.17 ([Rol03, p.10, Lemma A.5])**.** *Let B be a 3-ball. An automorphism of ∂B extends to a automorphism of B.*

Remark 2.18. Let T be a Conway tangle. The group $\mathrm{Mod}(\partial B_T \setminus \partial T) \overset{2.16}{=} \mathrm{PSL}_2(\mathbb{Z})$ is generated by

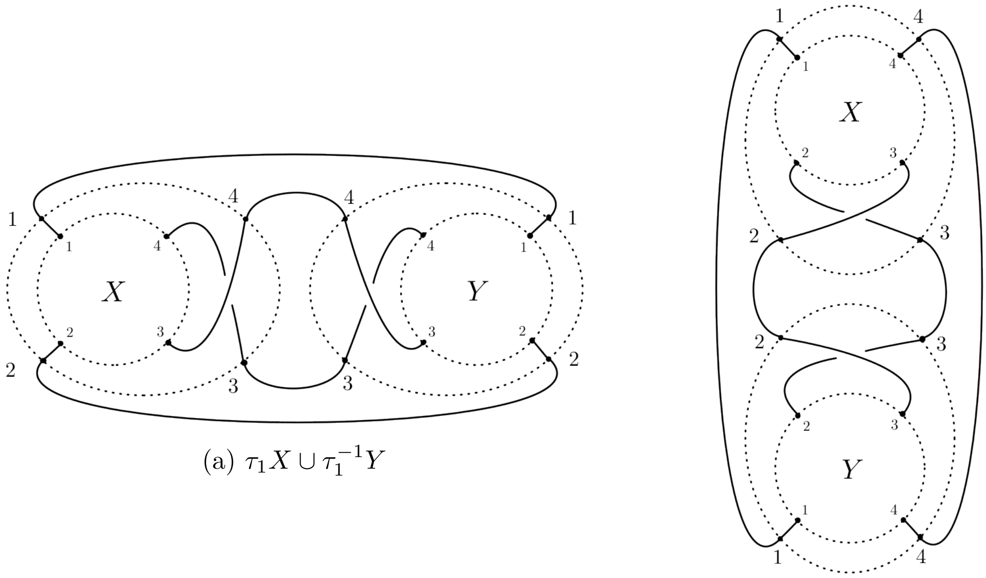
$$\tau_1 := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \tau_2 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

These to generators are actually *Dehn twists* of $S^2_4(T)$ which act on the parametrization $S \subset B_T$. Through this they also act on T by the half-twists illustrated in figure 10.

Another way think about this is fact 2.17. The two automorphism τ_1 and τ_2 extend to automorphisms of B_T and have same effect as above, but in this case we have to ignore the parametrization.

For an element $\tau \in \mathrm{Mod}(S^2_4(T))$ we denote the resulting Conway tangle by τT.

Definition 2.19 (Mirror)**.** Let T be an (oriented) Conway tangle. Let $\mathrm{m}(T)$ be the Conway tangle obtained by reversing the orientation of B_T, while preserving the labelling (and orientation of T). We call $\mathrm{m}(T)$ the **mirror of** T. A diagram of $\mathrm{m}(T)$ is obtained from one of T by changing all crossings.



(a) $\tau_1 X \cup \tau_1^{-1} Y$

(b) $\tau_2 X \cup \tau_2^{-1} Y$

Figure 11: Half-twist on unions

Remark 2.20 (Transformation of unions)**.** Let $L = X \cup Y$ be the oriented union of two Conway tangles. If we think about T as glued together along a single sphere $S :=$ $\partial B_X \cong \partial B_Y$, we can observe the action of $\mathrm{Mod}(S_4^2)$ on $S \setminus \partial X = S \setminus \partial Y$ and the auxiliary parametrizations of X and Y. It is clear, that the link type L is preserved under the action, but the question is, how the action differs for X and Y. For the generators $\tau \in \{\tau_1, \tau_2\}$ we can check that

$$X \cup Y = \tau^{-1} X \cup \tau Y = \tau X \cup \tau^{-1} Y$$

as illustrated in figure 11. If we define for a word (mind that we do not have commutativity, i.e. the order in the product matters)

$$A := \prod_{j=1}^{n} \tau_j^{a_j}$$

with $\tau_j \in \{\tau_1, \tau_2\}$ and $a_j \in \mathbb{Z}$ the notion

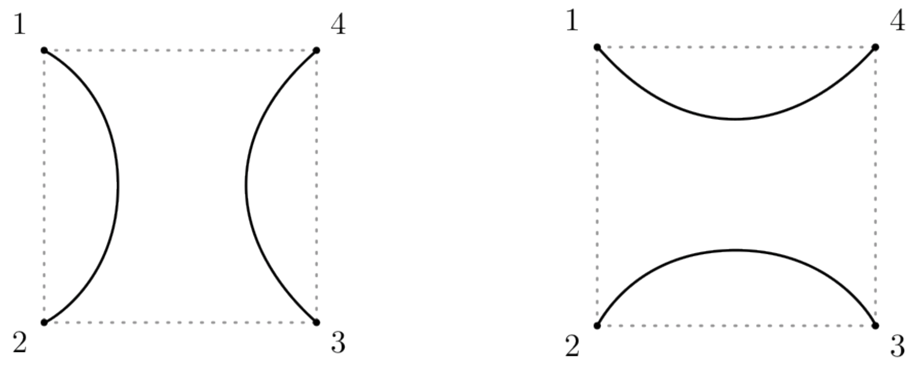
$$A^{\mathrm{m}} := \prod_{j=1}^{n} \tau_j^{-a_j}$$

then

$$X \cup Y = A^{\mathrm{m}} X \cup AY = AX \cup A^{\mathrm{m}} Y$$

as oriented links. Now think about $A^{\mathrm{m}} X$ diagrammatically: If we mirror $A^{\mathrm{m}} X$ it we get $A\,\mathrm{m}(X)$, hence

$$\mathrm{m}(A^{\mathrm{m}} X) = A\,\mathrm{m}(X).$$



(a) Q_∞ (b) Q_0

Figure 12: Trivial rational tangles

2.3 Rational tangles

We now have the prior knowledge to be able to introduce rational tangles and understand their classification.

Definition 2.21 (Rational tangle)**.** A **rational tangle** is a Conway tangle that is trivial as a 2-tangle.

Example 2.22. The easiest examples for rational tangles are the two *trivial* rational tangles Q_0 and Q_∞ depicted in figure 12 (the notation will become clear shortly). In fact, they are the elementary building blocks for all other rational tangles, as Conway's algorithm shows. Later, they become very important in the calculation of invariants due to their high symmetry.

Example 2.23. The (2, −3)-pretzel tangle $T_{2,-3}$ from figure 5 is not trivial (2.44) hence there are non-rational tangles.

Definition 2.24 ($\mathbb{Q}\mathrm{P}^1$)**.** We define

$$\mathbb{Q}\mathrm{P}^1 \coloneqq \mathbb{Q}^2/\sim$$

where $x \sim y \iff \exists_{\lambda\in\mathbb{Q}} : x = \lambda y$ for $x, y \in \mathbb{Q}^2$. Elements of $\mathbb{Q}\mathrm{P}^1$ are specified by the (reduced) slope of a line in $\mathbb{Q}^2$, hence

$$\mathbb{Q}\mathrm{P}^1 = \mathbb{Q} \cup \{\infty\}$$

where we use the convention

$$\infty = \frac{1}{0}.$$

Frequently we will use continued fractions $[a_1, \ldots, a_n]$, $n \in \mathbb{N}$, $a_1 \in \mathbb{Z}$, $a_i \in \mathbb{Z} \setminus \{0\}$ for $i \in \{2, \ldots, n\}$ to describe elements in $\mathbb{Q}\mathrm{P}^1$. As per usual the continued fractions evaluate according to

$$[a_1, \ldots, a_n] = a_1 + [a_2, \ldots, a_n]^{-1},$$

where we use the convention, that the empty continued fraction evaluates to infinity, i.e.

$$[\,] \coloneqq \infty \in \mathbb{Q}\mathrm{P}^1.$$

Definition 2.25 (Conway's Algorithm)**.** Given a continued fraction $C \coloneqq [a_1, \ldots, a_n]$ we define a rational tangle Q_C by

$$Q_C \coloneqq \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

As τ_1 and τ_2 can be thought of as automorphisms of the 3-ball (2.17) and Q_0, Q_∞ are trivial, the resulting Conway tangle Q_C is still trivial and hence a rational tangle.

If we start with the diagrams in figure 12 and apply the halft-twists from figure 10, we get the so called **twist-box diagram** of Q_C (the same as in [Cro+04, p.189, section 8.1]). The crossings resulting from a single $\tau_i^{a_j}$ in the definition are called a **twist-box**. For example, the tangle diagrams in figures 5a, 5b and 5c are the twist-box diagram for [ ], [3] and [2, 2, 2].

This gives us a way to associate with a number in $\mathbb{Q}\mathrm{P}^1$ an a priori multiple number of rational tangles, depending on the chosen continued fraction. On the other hand lemma 2.31 shows that we can carry the open components of a rational tangle via an isotopy to its border. Hence we can lift the *leading string* (as Conway said), i.e. the open component connected to the distinguished tangle end, up the two-fold toroidal cover. This gives us an embedding $K : S^1 \to T^2$ or in other words a torus knot K. By [Rol03, p.19, Theorem C.2] we can associate with K two coprime integers p and q hence a rational number. Mind that we could also use the secondary open component and that we use that K cannot be null-homologous, because the cover is branched at the tangle ends. We now have a connection between rational tangles and rational numbers in both direction and Conway showed, that this connection is as nice as it can get, thereby justifying the notion of rational tangle.

Theorem 2.26 (Classification of rational tangles. [Con70])**.** *Two rational tangles are isotopic if and only if they have the same fraction.*

For a nice self-contained combinatorial proof see the work [KL04] by Louis Kauffman and Sofia Lambropoulou.

Remark 2.27. By the last theorem Conway's algorithm gives rise to the bijection

$$\begin{array}{rcl} \mathbb{Q}\mathrm{P}^1 & \to & \{\text{rational tangles}\} \\ s & \mapsto & Q_s \end{array}$$

which finally makes the notation Q_s for $s \in \mathbb{Q}\mathrm{P}^1$ clear. Mind that the four 2-tangles in figure 5 are only correctly notated if we label the tangles end in the diagrams counter-clockwise starting at the upper left end.

Corollary 2.28. *Rational tangles are split.*

Proof. The tangles Q_∞ and Q_0 are clearly split. By Conway's algorithm 2.25 every other rational tangle results from applying an automorphism $\tau \in \mathrm{Mod}(S_4^2)$. A non-trivial compressing disk D of Q_∞ (or Q_0) then becomes a non-trivial compressing disk τD of τQ_∞ (or τQ_0). □

Corollary 2.29. *For* $p/q \in \mathbb{Q}\mathrm{P}^1$ *we have* $\mathrm{m}(Q_{p/q}) = Q_{-p/q}$.

Proof. Given a continued fraction $p/q = [a_1, \ldots, a_n]$ we know from Conway's algorithm 2.25 that

$$Q_{p/q} = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

Diagrammatically we see that

$$\mathrm{m}(Q_{p/q}) = \begin{cases} \tau_1^{-a_1}\tau_2^{-a_2}\tau_1^{-a_3}\cdots\tau_2^{-a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{-a_1}\tau_2^{-a_2}\tau_1^{-a_3}\cdots\tau_1^{-a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

And the continued fraction $[-a_1, \ldots, -a_n]$ evaluates to $-p/q$. □

The following proposition examines the effect of the mapping class group on the boundary of a rational tangle and the consequences on its fraction. For this purpose bear in mind that we can always associate with $p/q \in \mathbb{Q}\mathrm{P}^1$ the element

$$\frac{p}{q} \sim \begin{bmatrix} q \\ p \end{bmatrix} \in \mathbb{Z}^2/\mathbb{Z}^\times$$

which of course also works the other way around (with possible reductions).

Proposition 2.30 (Transform rational tangles)**.** *For* $p/q \in \mathbb{Q}\mathrm{P}^1$ *let* $\tau \in \mathrm{Mod}(S_4^2(Q_{p/q}))$, *then*

$$\tau Q_{p/q} = Q_{\tau\cdot\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]}.$$

In particular, $\tau Q_{p/q}$ *is still rational.*

Proof. We proof the statement for the generators $\tau_1^{\pm 1}, \tau_2^{\pm 1} \in \mathrm{Mod}(S_4^2)$ which then shows the claim by associativity. Let $p/q = [a_1, \ldots, a_n]$ be a continued fraction. Using Conway's algorithm 2.25 we see that τ_1 changes the continued fraction to

$$[a_1 + 1, \ldots, a_n] = \frac{p}{q} + 1 = \frac{p+q}{q}$$

which equals

$$\tau_1 \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} q \\ q+p \end{bmatrix} \sim \frac{p+q}{q}.$$

In similar manner τ_2 changes the continued fraction to

$$[0, 1, a_1, \ldots, a_n] = \frac{1}{\frac{q}{p}+1} = \frac{p}{q+p}$$

which equals

$$\tau_2 \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} q+p \\ p \end{bmatrix} \sim \frac{p}{q+p}.$$

Furthermore, τ_1^{-1} does

$$[a_1 - 1, \ldots, a_n] = \frac{p}{q} - 1 = \frac{p-q}{q}$$

which equals

$$\tau_1 \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} q \\ -q+p \end{bmatrix} \sim \frac{p-q}{q}.$$

Lastly, τ_2^{-1} does

$$[0, -1, a_1, \ldots, a_n] = \frac{1}{\frac{q}{p}-1} = \frac{p}{q-p}$$

which equals

$$\tau_2 \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} q-p \\ p \end{bmatrix} \sim \frac{p}{q-p}.$$

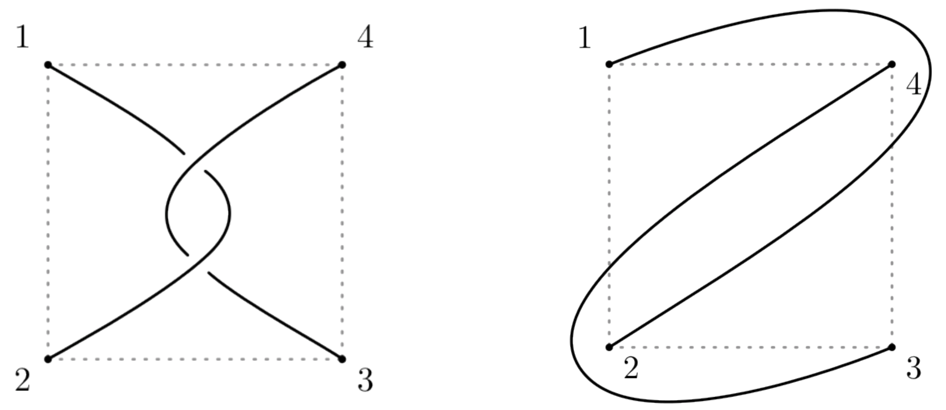
□

The next lemma was already used in the classification of rational tangles and gives a very important characterisation of trivial n-tangles.

Lemma 2.31 ([Cro+04, Lemma 4.10.2])**.** *Let T be a n-tangle. Then T is trivial if and only if there is an isotopy of B_T keeping ∂T fixed which carries the open components of T to ∂B_T.*

This lemma basically says, that each trivial n-tangle has a diagram without any crossing. One way to get such a diagram is to look at the **standard diagram** of rational links as used in [Hos19, Figure 3]. For the rational tangle $Q_{1/2}$ this isotopy is illustrated in figure 13

Remark 2.32 (Strand lifting)**.** Let $p/q \in \mathbb{Q}\mathrm{P}^1$ and $T := Q_{p/q}$ isotoped via lemma 2.31 such that the open components lie on the border ∂B_T. After removing the tangle ends ∂T, we get two open strands $T(I \sqcup I) \setminus \partial T$ lying on the four punctured sphere $\partial B_T \setminus \partial T$ each connecting two punctures.



(a) Twist-box diagram of $Q_{1/2}$ (b) Crossing-less diagram of $Q_{1/2}$

Figure 13: Open components isotoped to the border

Then *the lift of any of these open strands along η is homotopic to an (open) straight line segment in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ of slope $p/q \in \mathbb{Q}\mathrm{P}^1$.* For the trivial rational tangles Q_∞ and Q_0 (Figure 12) we can easily see this by isotoping the open open components to the sites a, b respectively c, d. The sites then get lifted by assumption to straight line segments of the wanted slope (Figure 9). For an arbitrary $Q_{p/q}$ we would use proposition 2.30 and the fact that the action of the mapping class groups $\mathrm{Mod}(S_4^2)$ is induced by the (linear) action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Remark 2.33 (Connectivity)**.** It is often useful to know which tangle ends are connected by the open components. In the case of a rational tangles $Q_{p/q}$ we can actually read this off the fraction $p/q \in \mathbb{Q}\mathrm{P}^1$. We use last remark 2.32 to get open strands on the border connecting our tangle ends which lift along η to open straight line segments in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ of slope p/q. From this slope we can now easily determine the connectivity in the covering space:

$$1 \text{ is connected to } \begin{cases} 2, & \text{if } p \text{ is odd, } q \text{ is even,} \\ 3, & \text{if } p \text{ is odd, } q \text{ is odd,} \\ 4, & \text{if } p \text{ is even, } q \text{ is odd.} \end{cases}$$

2.4 Rational links

With rational tangles and the methods of the last section, we have now classified trivial 2-tangles. This inevitably directs our attention to links that look locally like these rational tangles and how we can describe them globally.

Definition 2.34 (Rational link)**.** A link is called **rational** if it equivalent to the union of two rational tangles.

Definition 2.35 (Numerator, Denominator)**.** Let T be a rational tangle. The **numerator** $N(T)$ of T is the link $Q_0 \cup T$. The **denominator** $D(T)$ of T is the link $Q_\infty \cup T$.

In addition we use the notation

$$N(s) := N(Q_s) \qquad D(s) := D(Q_s)$$

for $s \in \mathbb{Q}\mathrm{P}^1$.

Remark 2.36. By definition the numerator and denominator (2.35) of rational tangles are rational links. As rational tangles have no closed components, it is clear that rational links can at most have two components. However, not all links with two or less components are rational as we can see in the *Kinoshita-Terasaka knot* (11n42) [Cro+04, p. 97]. Later (2.49) we will see that rational links are alternating and hence get a criterion for non-rationality. Yet, all knots with crossing number at most seven are rational as well as all two-component links with crossing number at most six [Cro+04, p. 211]. Classification can be further done by examining repeated partial sums of rational tangles, which are called *algebraic* tangles, and then plugging in algebraic tangles as vertices of 4-valent graphs with no 2-gons (*basic polyhedra*). Conway used this technique in the computation of the link table in [Con70].

The following notion was introduced by Horst Schubert (see [Sch56]).

Definition 2.37 (Bridge presentation)**.** Let $\mathbb{R}^3_h := \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ be the closed upper half-space of $\mathbb{R}^3$, $\pi : \mathbb{R}^3_h \to \partial\mathbb{R}^3_h = \mathbb{R}^2$ be the canonical projection from the upper half-space onto the plane and let $L \subset \mathbb{R}^3_h$ be a link. If the image

$$\pi(L \cap (\mathbb{R}^3_h \setminus \mathbb{R}^2)) \subset \mathbb{R}^2,$$

i.e. the projection of the subset of L which does not lie in the plane, consists only of disjoint straight line segments, we say that L is in **bridge presentation**. The components of $L \cap (\mathbb{R}^3_h \setminus \mathbb{R}^2)$ are the **bridges** of the particular bridge presentation.

The minimal number of bridges required for a given link to be in bridge presentation is called the **bridge number** of this link.

Remark 2.38. For a given link we can look at an arbitrary link diagram of it and raise the upper strand of each crossing into the upper half-space to get a bridge presentation.

Lemma 2.39. *One-bridge links are trivial.*

Proof. Let $L \subset \mathbb{R}^3_h$ be an n-component one-bridge link. As L is locally flat the embedding $L \cap \mathbb{R}^2 \hookrightarrow \mathbb{R}^2$ is locally flat, hence we can find a tubular neighbourhood U of $L \cap \mathbb{R}^2$ in $\mathbb{R}^2$. The bridge $L \cap (\mathbb{R}^3_h \setminus \mathbb{R}^2)$ can then be isotoped to lie in U. Therefore L is equivalent to a link $f : \bigsqcup_n S^1 \hookrightarrow \mathbb{R}^2$. As f is an embedding $\operatorname{im} f$ must be the disjoint union of n embedded loops in $\mathbb{R}^2$ which are themselves equivalent to the unknot by Schoenflies' theorem. □

Theorem 2.40 ([Cro+04, Theorem 4.10.3])**.** *A rational link has a two-bridge presentation.*

Corollary 2.41. *Each rational link is the numerator of a rational tangle.*

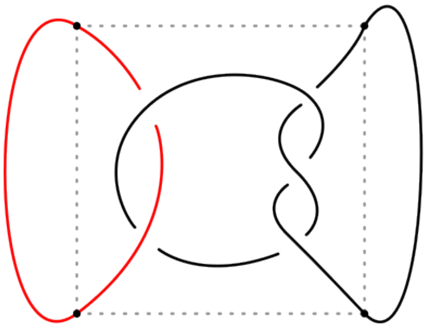


Figure 14: The link $Q_\infty \cup T_{2,-3}$

Proof. A rational link L has a two-bridge presentation by Theorem 2.40. We can then embed a 3-ball B in $\mathbb{R}^3$ such that B intersects L transversely in four points and $L \cap \mathbb{R}^2 \subset B$, but neither of the two bridges lies entirely in B. We then parametrize this tangle $L \cap B \hookrightarrow B$ such that one bridge connects the tangle ends 1 with 4 and the other 2 with 3. □

See lemma 2.51 on how to compute this numerator form.

Corollary 2.42. *The only split rational link is the unlink.*

Proof. Let L be a split rational link (hence a two-component link). By corollary 2.41 the link L is the numerator of a rational tangle, i.e.

$$L = Q_0 \cup Q$$

for a rational tangle Q. Because Q has no closed components the open components of Q_0 must belong to different components of L. As L is split the two-bridge presentation resulting from lemma 2.31 is split (we think about ∂Q as $\mathbb{R}^2 \cup \{\infty\}$). Hence L is the disjoint union of two one-bridge knots. By lemma 2.39 these are trivial. □

As all rational tangles are split (2.28), this shows that the converse statement of remark 2.15 is false. For (counter-)example, we could look at the Hopf link $Q_0 \cup Q_2$.

Remark 2.43 (Two-bridge links are rational)**.** Given a two-bridge link L. We think of S^3 as $\mathbb{R}^3 \cup \{\infty\}$ and $\mathbb{R}^2 \cup \{\infty\}$ as embedded 2-sphere. For a suitable parametrization of the four bridge ends, the two hemispheres can be thought of as Conway tangles. One is isomorphic to Q_0 and the other one has its open components already isotoped into its border, hence is rational by lemma 2.31. Therefore L is a rational link.

Remark 2.44 (Non-trivial tangles)**.** We now have the equipment to prove that there exist non-trivial tangles, for example the (2, -3)-pretzel tangle.

Assume that $T_{2,-3}$ is trivial. The the union

$$L \coloneqq Q_\infty \cup T_{2,-3}$$

shown in figure 14 is by definition a two-component rational link. By theorem 2.40 L has a two-bridge presentation. As L is non-split (by "flipping" the left component we get an alternating non-split diagram), the two bridges must belong to different components. By simply forgetting about the "left" component (coloured in red above), the remaining knot is the left-handed trefoil knot with a apparent one-bridge presentation. By lemma 2.39 the left-handed trefoil is therefore trivial. This is a contradiction to its non-triviality (first shown by Heinrich Tietze in 1908 [Tie08]).

Here we shifted the problem from non-triviality of a tangle to non-triviality of a link, which we can show then with some link invariant. To show triviality of a tangle T we would have to find an element of $\tau \in \mathrm{Mod}(S^2_4(T))$ such that τT is trivial or use some topological argument with lemma 2.31. However, later we will learn about a computable invariant that detects rational tangles.

Definition 2.45 (Fraction of a rational links)**.** If L is a rational link and ${}^p/_q \in \mathbb{Q}\mathrm{P}^1$ such that

$$L = N({}^p/_q),$$

we call ${}^p/_q$ a **fraction of** L.

By corollary 2.41 every rational link has at least one fraction. From easy examples like $N(1) = N({}^1/_2) = U_1$ we see that there are in general multiple fractions of the same rational link. In fact there are always infinitely many such fractions, as the next theorem by Schubert shows.

Theorem 2.46 ([Sch56])**.** *Let ${}^{p_1}/_{q_1}, {}^{p_2}/_{q_2} \in \mathbb{Q}\mathrm{P}^1$. It holds that $N({}^{p_1}/_{q_1})$ and $N({}^{p_2}/_{q_2})$ are equivalent if and only if*

1. $p_1 = p_2$ *and*
2. *either* $q_1 \equiv q_2 \pmod{p_1}$ *or* $q_1 q_2 \equiv 1 \pmod{p_1}$.

Corollary 2.47. *Let ${}^p/_q \in \mathbb{Q}\mathrm{P}^1$. If $N({}^p/_q)$ is split, then ${}^p/_q = 0$.*

Proof. By corollary 2.42 the link $N({}^p/_q)$ is the unlink, hence equivalent to $N(0)$. By theorem 2.46 we get that $p = 0$. □

Schubert also extended his theorem to the case of oriented rational links:

Theorem 2.48 ([Sch56])**.** *Suppose that equally oriented rational tangles with fractions ${}^{p_1}/_{q_1}$ and ${}^{p_2}/_{q_2}$ are given with q_1 and q_2 odd. It holds that the oriented $N({}^{p_1}/_{q_1})$ and $N({}^{p_2}/_{q_2})$ are equivalent if and only if*

1. $p_1 = p_2$ *and*
2. *either* $q_1 \equiv q_2 \pmod{2p_1}$ *or* $q_1 q_2 \equiv 1 \pmod{2p_1}$.

See [KL02] for elegant combinatorial proofs of these two theorems.

Theorem 2.49 ([Cro+04, p. 205, Theorem 8.7.1])**.** *Rational links are alternating.*

Proof. Let L be a rational link with fraction $p/q \in \mathbb{Q}\mathrm{P}^1$. There is a continued fraction expansion of p/q in which all the coefficients have the same sign. The corresponding tangle diagram resulting from Conway's algorithm 2.25 applying the half-twists from remark 2.18 has an alternating numerator. □

Lemma 2.50. *Let x/y, $p/q \in \mathbb{Q}\mathrm{P}^1$ and*

$$L \coloneqq Q_{x/y} \cup Q_{p/q}$$

be a rational link. For $\tau \in \mathrm{Mod}(S_4^2)$ holds that

$$L = \mathrm{m}\, Q_{\tau\left[\begin{smallmatrix}-y\\ x\end{smallmatrix}\right]} \cup Q_{\tau\left[\begin{smallmatrix}q\\ p\end{smallmatrix}\right]}.$$

Proof. From remark 2.20 we get that

$$L = \tau^{\mathrm{m}} Q_{x/y} \cup \tau Q_{p/q} = \mathrm{m}(\tau\, \mathrm{m}(Q_{x/y})) \cup \tau Q_{p/q}.$$

The claim follows from corollary 2.29 and proposition 2.30. □

Lemma 2.51 (Numeratorize a rational link)**.** *Let x/y, $p/q \in \mathbb{Q}\mathrm{P}^1$ and*

$$L \coloneqq Q_{x/y} \cup Q_{p/q}$$

be a rational link. As x and y are coprime we find $a, b \in \mathbb{Z}$ such that

$$ax + by = -1$$

for which we define

$$\tau \coloneqq \begin{bmatrix} -b & a \\ x & y \end{bmatrix} \in \mathrm{Mod}(S_4^2).$$

Then the link

$$Q_0 \cup Q_{\tau\left[\begin{smallmatrix}q\\ p\end{smallmatrix}\right]}$$

is equivalent to L.

Proof. First, observe that $\det \tau = -by - ax = -(ax + by) = 1$, hence $\tau \in \mathrm{PSL}_2(\mathbb{Z})$. Next, we use lemma 2.50 with τ and get

$$L = \mathrm{m}\, Q_{\tau\left[\begin{smallmatrix}-y\\ x\end{smallmatrix}\right]} \cup Q_{\tau\left[\begin{smallmatrix}q\\ p\end{smallmatrix}\right]} = \mathrm{m}\, Q_{\left[\begin{smallmatrix}-1\\ 0\end{smallmatrix}\right]} \cup Q_{\tau\left[\begin{smallmatrix}q\\ p\end{smallmatrix}\right]} = \mathrm{m}\, Q_0 \cup Q_{\tau\left[\begin{smallmatrix}q\\ p\end{smallmatrix}\right]}$$

and clearly

$$\mathrm{m}\, Q_0 = Q_0.$$

□

3 The genus of a link

3.1 Seifert surfaces

The following notions were introduced by Herbert Seifert in [Sei33] (his habilitation thesis) and [Sei34].

Definition 3.1 (Seifert surface)**.** Let L be an oriented link. A **Seifert surface** of L is a compact, oriented and connected surface with boundary embedded in S^3 such that L is its oriented boundary.

Definition 3.2 (Genus of a Link)**.** Let L be a oriented link. The (Seifert) **genus** $g(L)$ of L is the minimal genus over all Seifert surfaces of L.

Example 3.3 (Unlinks)**.** We can find an ambient isotopy such that the oriented unknot U_1 is the oriented boundary of the closed unit disk $B^2 \subset \mathbb{R}^2 \subset S^3$. The oriented closed unit disk is an oriented 2-sphere with a open disk removed, and hence has the same genus as the 2-sphere by definition. Therefore, the genus of the unknot is zero. By the same argument with multiple open disks removed from a 2-sphere we get that

$$g(U_n) = 0$$

for all $n \in \mathbb{N}_{>0}$.

Remark 3.4 (Unknot detection)**.** If K is a knot with $g(K) = 0$ we know that K must be the boundary of a oriented embedded disk F. By contracting F we get that K must already be the unknot. Hence the genus for knots detects the unknot.

The genus of a link is well-defined, as we can construct a Seifert surface from every link diagram by using Seifert's algorithm (next section). However, regardless of the link diagram Seifert's algorithm is not always capable of constructing a Seifert surface with minimal genus. Therefore the genus of a link is quite hard to compute from the definition. Later we will see better ways to compute the genus via Heegaard Floer invariants.

Remark 3.5 (Euler characteristic)**.** A Seifert surface F can be triangulated ([Cro+04, Fact 2.7.1, p. 41]. As F is also compact this triangulation can be assumed to be finite. Out of such a finite triangulation with V vertices, E edges and T triangles we get the Euler characteristic

$$\chi(F) := V - E + T$$

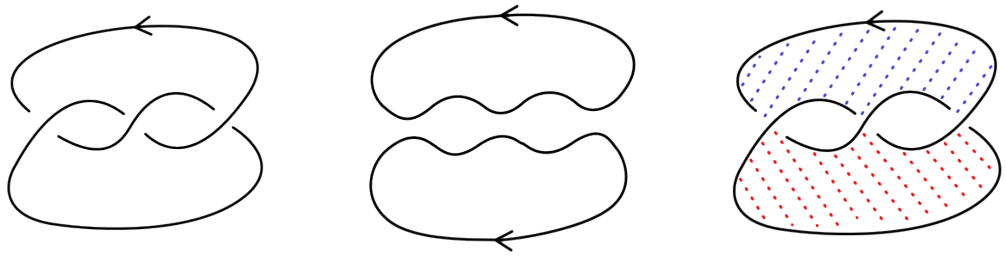
of our surface F. This Euler characteristic is related to the genus $g(F)$ of F by the following formula

$$2g(F) = 2 - \chi(F) - |\partial F|$$

where $|\partial F|$ denotes the number of boundary components of F

If we define the **Euler characteristic** $\chi(L)$ **of an link** L to be the maximum Euler characteristic over all Seifert surfaces, we get the following equation:

$$g(L) = \frac{2 - \chi(L) - \mu(L)}{2}$$



(a) Trefoil diagram D (b) Seifert circles of D (c) Projection surface F

Figure 15: Seifert's algorithm for the left-handed trefoil

3.2 Seifert's Algorithm

This algorithm was introduced by Seifert in [Sei34] and can be used to show that every oriented link has at least one Seifert surface, i.e. that the genus of a link is always a natural number and not infinity.

Theorem 3.6 ([Cro+04, Theorem 5.1.1])**.** *Every oriented link has a Seifert surface.*

We omit the (not to hard) algorithm but it can be looked up in the proof of [Cro+04, Theorem 5.1.1].

Definition 3.7 (Projection surface)**.** Let D be a oriented link diagram. We call the surface constructed by Seifert's algorithm the **projection surface** of D. From the algorithm we get a property of the diagram D, namely the number of **Seifert circles** that occur during execution.

Proposition 3.8 (cf. [Cro+04, Theorem 5.1.2])**.** *The projection surface F of some oriented link diagram with c crossings, s Seifert circles and μ components has the Euler characteristic*

$$\chi(F) = s - c$$

and therefore the genus

$$g(F) = \frac{2 - s + c - \mu}{2}.$$

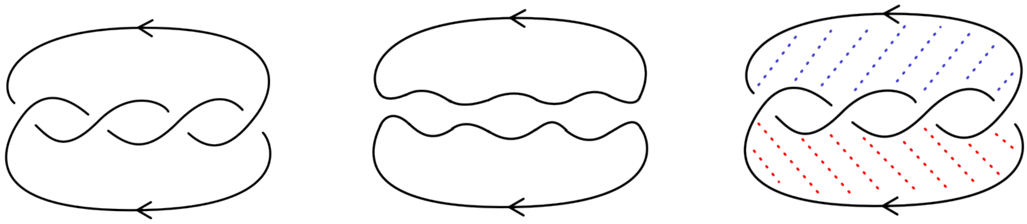
Example 3.9. Using Seifert's algorithm 3.6 we construct the projection surface F of a diagram D of the left-handed trefoil knot illustrated in figure 15. The diagram has three crossings and two Seifert circles. The two sides of the surface are coloured red and blue for distinction. By proposition 3.8 we get that

$$g(F) = \frac{2 - 2 + 3 - 1}{2} = 1.$$

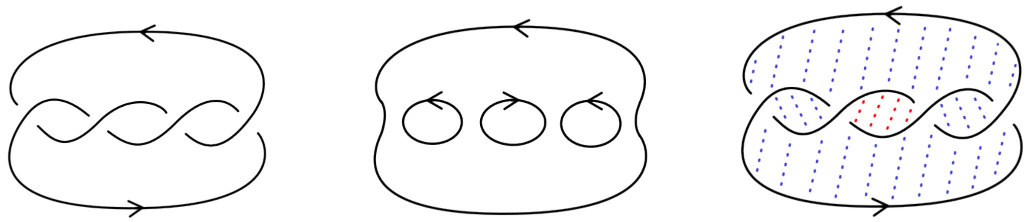
As the left-handed trefoil is a non-trivial knot, we get by remark 3.4 that $g(K) \neq 0$ and hence

$$0 < g(K) \overset{\text{Def.}}{\leq} g(F) = 1.$$

This shows that $g(K) = 1$.



(a) D_1 (b) Seifert circles of D_1 (c) F_1



(d) D_2 (e) Seifert circles of D_2 (f) F_2

Figure 16: Seifert's Algorithm on $N(Q_4)$

The next statement was proven independently by Kunio Murasugi in 1958 [Mur58] and Richard Crowell in 1959 [Cro59]. Remember that a diagram is *reduced*, if it is non-split and neither resolution of any crossing results in a split diagram.

Proposition 3.10 ([AK13, Theorem 1.1])**.** *The projection surface of a reduced alternating oriented link diagram has minimal genus.*

Caveat 3.11 (Orientation matters)**.** We can use the last proposition 3.10 to verify that the genus of a link actually depends on the chosen orientation. Let L_1 and L_2 be the two orientations of $N(Q_4)$ depicted in the diagrams D_1 and D_2 in figure 16. From proposition 3.8 we get (maybe after a index swap) that

$$g(F_1) = 1 \quad \text{and} \quad g(F_2) = 0$$

and from proposition 3.10 that

$$g(L_1) = 1 \quad \text{and} \quad g(L_2) = 0.$$

3.3 Genus of rational links

Lemma 3.12 (Even continued fraction)**.** *Let* $p/q \in \mathbb{Q}\mathrm{P}^1$ *with* $p + q \equiv 1 \pmod 2$*. Then there are unique* $n \in \mathbb{N}$ *and* $a_1 \in 2\mathbb{Z}$ *and* $a_2, \ldots, a_n \in 2\mathbb{Z} \setminus \{0\}$ *such that*

$$p/q = [a_1, \ldots, a_n].$$

with the additional property, that for all $i \in \{2, \dots, n\}$

$$[a_i, \dots, a_n]^{-1} \in (-1, 1).$$

We call this the ***even continued fraction*** *of p/q.*

Proof. The number p/q cannot be an odd integer by assumption, hence there is a unique $a_1 \in 2\mathbb{Z}$ such that

$$s := p/q - a_1 \in (-1, 1).$$

If $s \neq 0$ we repeat the argument for $s^{-1} \in \mathbb{Q}$. Observe that the parity assumption still holds as either q or $p - q \cdot a_a$ is even. As $s^{-1} \in \mathbb{Q} \setminus (-1, 1)$ the next coefficient cannot be zero. The algorithm terminates because after every step the absolute value of the denominator of our fraction becomes strictly smaller. The additional property holds simply by choice of the coefficients. □

Remark 3.13 (Even continued fraction of links)**.** Let L be a rational Link with fraction p/q. By Theorem 2.46 we know that

$$N(p/q) = N(p/(q+p))$$

hence we can always assume that $p + q \equiv 1 \pmod 2$ for the fraction p/q of a rational Link. Therefore we can talk about **even continued fractions of links**.

Remark 3.14. Let $p/q \in \mathbb{Q}\mathrm{P}^1$ with $p + q \equiv 1 \pmod 2$ and let $p/q = [a_1, \dots, a_n]$ be an even continued fraction. By evaluating the continued fraction "backwards", i.e. by first calculating $a_{n-1} + 1/a_n$ and then going further up, we see that the length n determines the parity of p and q. More precisely we have

$$\begin{aligned} n \text{ even} \quad &\Longleftrightarrow \quad p \text{ odd}, q \text{ even}, \\ n \text{ odd} \quad &\Longleftrightarrow \quad p \text{ even}, q \text{ odd}. \end{aligned}$$

With remark 2.33 we get

$$\begin{aligned} n \text{ even} \quad &\Longleftrightarrow \quad N(p/q) \text{ is a knot}, \\ n \text{ odd} \quad &\Longleftrightarrow \quad N(p/q) \text{ is a link}. \end{aligned}$$

Definition 3.15 (Cyclical orientation)**.** We call an oriented Conway tangle T **cyclically oriented** if the tangle ends are alternately pointing inwards and outwards (along the fixed circle $S \subset B_T$).

Let L be an oriented rational link. We call an even continued fraction $C := [a_1, \dots, a_n]$ of L **cyclical**, if for

$$L = Q_0 \cup Q_C$$

the oriented rational tangle Q_0 is cyclically oriented.

Remark 3.16 (Cyclical even continued fraction of links)**.** By remark 3.13 each rational link L has a fraction $p/q \in \mathbb{Q}\mathrm{P}^1$, $p + q \equiv 1 \pmod 1$ such that

$$L = Q_0 \cup Q_{p/q}.$$

If we fix an orientation on L then Q_0 (and $Q_{p/q}$) inherit an orientation. If Q_0 is cyclically oriented we have found a cyclical even continued fraction. This is always the case when L is a knot as then 1 is connected to 2 by remark 2.33. Otherwise L is a link, hence we know that 1 is connected to 4 and therefore q is odd. Now we apply τ_2 as in remark 2.20 and get

$$L = \tau_2^{-1} Q_0 \cup \tau_2 Q_{p/q}$$

where $\tau_2^{-1} Q_0$ is equivalent to Q_0 but cyclically oriented. By lemma 2.30 we get that

$$\tau_2 Q_{p/q} = Q_{\tau_2 \left[\begin{smallmatrix} q \\ p \end{smallmatrix} \right]} = Q_{\frac{p}{q+p}}.$$

As $p + (q + p) \equiv q \equiv 1 \pmod 2$, the fraction $\frac{p}{q+p}$ has an even continued fraction. Hence each oriented rational link has a cyclical even continued fraction.

Proposition 3.17 (cf. [Cro+04, Corollary 8.7.5])**.** *Let L be an oriented rational link with cyclical even continued fraction $[a_1, \ldots, a_n]$. Then it holds*

$$g(L) = \begin{cases} 1/2\, n, & \text{if } n \text{ is even} \\ 1/2\,(n-1), & \text{if } n \text{ is odd} \end{cases}$$

if $a_1 \neq 0$ or $n \leq 1$. Otherwise

$$g(L) = \left\{ \begin{matrix} 1/2\, n, & \text{if } n \text{ is even} \\ 1/2\,(n-1), & \text{if } n \text{ is odd} \end{matrix} \right\} - 1$$

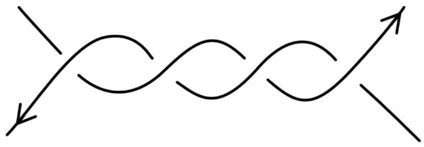
holds.

Proof. If $n = 0$ then $L = U_1$ and if $n = 1$, $a_1 = 0$ then $L = U_2$, hence the formula holds.
Let $p/q = [a_1, \ldots, a_n]$ and $Q_{p/q}$ be oriented cyclically such that

$$L = Q_0 \cup Q_{p/q}.$$

If $n > 0$, $a_1 \neq 0$ we are in the case of the normalization $-p < q < p$ which is one of two used in [Cro+04, Corollary 8.7.5]. Note that Cromwell assumes cyclical orientation for all rational tangles. [Cro+04, Definition 7.8.1]
If $n > 1$, $a_1 = 0$ we have the following equations

$$\begin{aligned} p/q &= [0, a_2, \ldots, a_n] \\ -q/p &= [-a_2, -a_3, \ldots, -a_n] \end{aligned}$$


(a) A twist-box containing four twists

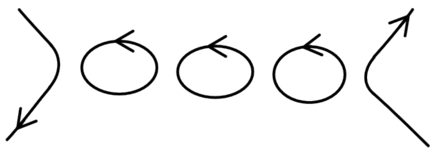

(b) Three (inner) Seifert circles

Figure 17: Seifert's algorithm on twist-boxes

and therefore

$$\frac{-q + a_2 p}{p} = [-a_2 + a_2,\, -a_3,\, \dots,\, -a_n] = [0,\, -a_3,\, \dots,\, -a_n]$$
$$\frac{p}{q - a_2 p} = [a_3,\, \dots,\, a_n] = (*).$$

Now the even continued fraction $(*)$ is either empty or the first coefficient is not zero. In both cases the length is $n-2$. Now remark 2.20 and proposition 2.30 show that

$$Q_0 \cup Q_{p/q} = \tau_2^{a_2} Q_0 \cup \tau_2^{-a_2} Q_{p/q} = Q_0 \cup Q_{\frac{p}{(q-a_2 p)}}$$

as oriented links, and because a_2 is even the tangles are still oriented cyclically. In this situation we can use the already proven formula. □

Remark 3.18. The proof of the formula [Cro+04, Corollary 8.7.5] works by computing the Alexander-Conway polynomial of the diagram given by Conway's algorithm 2.25 for the even continued fraction. The even coefficients make it possible to compute this polynomial in a systematic way [Cro+04, Theorem 8.7.4]. The prove needs that rational links are alternating 2.49. Furthermore this shows that the projection surface F of the diagram D build from the even continued fraction already has minimal genus:

The cyclical orientation and the even number of twists in each twist-box (the twists generated by a single τ_1^a or τ_2^a for $a \in \mathbb{Z}$) ensure that the number of Seifert circles of the diagram D is exactly

$$s = 1 + \sum_{i=1}^{n} (a_i - 1).$$

The reason for this is that the strands in each twist box run in opposite directions (exemplified in figure 17). Mind that the additional (outer) Seifert circle is the the one going through all twist-box ends and the arcs of Q_0. Hence proposition 3.8 shows that

$$g(F) = \frac{2 - s + \sum_{i=1}^{n} a_i - \mu(L)}{2}$$
$$= \frac{1 + n - \mu(L)}{2}$$

and notice that L is a knot if n is even and a two-component link if n is odd. This gives exactly the formula of proposition 3.17 for the case $a_1 \neq 0$ or $n \leq 1$.

For an alternative proof using completely different techniques see section 5.3.

Remark 3.19. Let $L = Q_s \cup Q_t$ be an oriented rational link and $s = [a_1, \dots, a_n]$ a continued fraction. By Conway's algorithm 2.25 we know

$$Q_s := \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}.$$

If we define

$$\pi := \left\{\begin{matrix} \tau_2^{a_n}\cdots\tau_1^{a_3}\tau_2^{a_2}\tau_1^{a_1}, & \text{if } n \text{ is even,} \\ \tau_1^{a_n}\cdots\tau_1^{a_3}\tau_2^{a_2}\tau_1^{a_1}, & \text{if } n \text{ is odd,} \end{matrix}\right\} \in \mathrm{Mod}(S_4^2)$$

then remark 2.20 and associativity imply

$$L = \pi^{\mathrm{m}} Q_s \cup \pi Q_t = \begin{cases} Q_\infty \cup \pi Q_t, & \text{if } n \text{ is even,} \\ Q_0 \cup \pi Q_t, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, because L is still oriented, it is clear that

$$\begin{aligned} \pi Q_t(\infty) &\quad \text{exists, if } n \text{ is even,} \\ \pi Q_t(0) &\quad \text{exists, if } n \text{ is odd.} \end{aligned}$$

Definition 3.20 (Rational closures)**.** Let T be a oriented Conway tangle and $s \in \mathbb{Q}\mathrm{P}^1$. If Q_{-s} is orientable such that

$$T(s) := Q_{-s} \cup T$$

is an oriented link, we call this link the **s-closure of T**.

Proposition 3.21. *Let Q_0 and $Q_{1/2}$ be oriented cyclically and $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0, \infty\}$. If $Q_0(p/q)$ exists with cyclical even continued fraction $[a_1, \dots, a_n]$, then the rational link $Q_{1/2}(p/q)$ has a cyclical even continued fraction*

$$\begin{aligned} [-2, a_1, \dots, a_n] &\quad \textit{if} \quad a_1 \neq 0, \\ [a_2 - 2, a_3, \dots, a_n] &\quad \textit{if} \quad a_1 = 0. \end{aligned}$$

Furthermore, $Q_{1/2}(p/q)$ has a cyclical even continued fraction

$$\begin{aligned} [0] &\quad \textit{if} \quad p/q = 0, \\ [-2] &\quad \textit{if} \quad p/q = \infty. \end{aligned}$$

Proof. We use remark 2.20 to apply the transformation

$$\tau_2^{-2} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \in \mathrm{Mod}(S_4^2)$$

to

$$Q_{-p/q} \cup Q_{1/2}$$

and get by proposition 2.30

$$\tau_2^2 Q_{-p/q} \cup \tau_2^{-2} Q_{1/2} = Q_{\frac{p}{q+2p}} \cup Q_\infty$$

where the rational tangles are still oriented cyclically as the exponent of τ_2 is even. We now apply the transformation

$$\tau_F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Mod}(S_4^2)$$

and get

$$\tau_F Q_{\frac{p}{q+2p}} \cup \tau_F^{-1} Q_\infty = Q_{\frac{-q-2p}{p}} \cup Q_0.$$

Mind that τ_F preserves the cyclical tangle orientation. Hence we only need to find an even continued fraction for $-q/p - 2$. Note that

$$-p/q = [a_1, \ldots, a_n]$$

by assumption.

Nothing happens in the following equation, but it helps to understand the following arguments.

$$-\frac{q}{p} = \frac{1}{-p/q} = \frac{1}{[a_1, \ldots, a_n]} = [0, a_1, \ldots, a_n]$$

If $a_1 \neq 0$ we get that

$$\frac{1}{[a_1, \ldots, a_n]} \in (-1, 1)$$

and hence

$$-q/p - 2 = [-2, a_1, \ldots, a_n]$$

is already the even continued fraction. If $a_1 = 0$ we have

$$-\frac{q}{p} = [0, 0, a_2, \ldots, a_n] = [a_2, \ldots, a_n]$$

and therefore

$$-q/p - 2 = [a_2 - 2, a_3, \ldots, a_n].$$

□

The next two corollaries basically follow from going very carefully through Propositions 3.17 and 3.21. They have been hinted to by the graphical visualizations figure 40 and 41 in appendix B.

Lemma 3.22. *Let Q_0 and $Q_{1/2}$ be oriented cyclically. Let $p/q \in \mathbb{QP}^1$ with p **even** and q **odd**. Then it holds*

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) \quad \textit{for} \quad p/q \in \{0\} \cup (1/3,\, 1)$$

and

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) - 1 \quad \textit{otherwise.}$$

Proof. Let $[a_1, \ldots, a_n]$ be a cyclical even continued fraction of $Q_0(p/q)$. As q is odd we know from Remark 3.14 that n must be odd. If $a_1 \neq 0$ we have by 3.21 that

$$[-2, a_1, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}(n-1) = \frac{1}{2}(n+1) - 1 \overset{3.17}{=} g(Q_{1/2}(p/q)) - 1.$$

If $a_1 = 0$ and $n = 1$ we have

$$g(Q_0(0)) = g(U_2) \overset{3.17}{=} \frac{1}{2}(n-1) = 0 \overset{3.21}{=} g(U_1) = g(Q_{1/2}(0))$$

therefore assume $n > 1$. If $a_2 \neq 2$ we have by 3.21 that

$$[\underbrace{a_2 - 2}_{\neq 0}, a_3, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}(n-1) - 1 \overset{3.17}{=} g(Q_{1/2}(p/q)) - 1.$$

If $a_2 = 2$ we have by 3.21 that

$$[0, a_3, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}(n-1) - 1 \overset{3.17}{=} g(Q_{1/2}(p/q)).$$

Interpreting these cases in rational numbers using

$$[0, 2, a_3, \ldots, a_n] = \frac{1}{2 + \underbrace{[a_3, \ldots, a_n]^{-1}}_{\in(-1,1)}}$$

we get the claim. □

Lemma 3.23. *Let Q_0 and $Q_{1/2}$ be oriented cyclically. Let $p/q \in \mathbb{QP}^1$ with p **odd** and q **even**. Then it holds*

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) + 1 \quad \textit{for} \quad p/q \in (1/3, 1) \setminus \{1/2\}$$

and

$$g(Q_0(p/q)) = g(Q_{1/2}(p/q)) \quad \textit{otherwise.}$$

Proof. Let $[a_1, \ldots, a_n]$ be a cyclical even continued fraction of $Q_0(p/q)$. As q is even we know from remark 3.14 that n must be even. If $n = 0$ we have

$$g(Q_0(\infty)) = g(U_1) \overset{3.17}{=} \frac{1}{2}n = 0 = \frac{1}{2}((n+1)-1) \overset{3.17}{=} g(Q_{1/2}(\infty))$$

hence assume $n > 0$. If $a_1 \neq 0$ we have by 3.21 that

$$[-2, a_1, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}n = \frac{1}{2}((n+1)-1) \overset{3.17}{=} g(Q_{1/2}(p/q)).$$

If $a_1 = 0$ and $a_2 \neq 2$ we have by 3.21 that

$$[\underbrace{a_2 - 2}_{\neq 0}, a_3, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}n - 1 = \frac{1}{2}((n-1)-1) \overset{3.17}{=} g(Q_{1/2}(p/q)) - 1.$$

If $a_2 = 2$ and $n = 2$ we have

$$g(Q_0(1/2)) \overset{3.17}{=} \frac{1}{2}n - 1 = 0 = \frac{1}{2}((n-1)-1) \overset{3.17}{=} g(Q_{1/2}(1/2))$$

therefore assume $n > 2$. Then we have by 3.21 that

$$[0, a_3, \ldots, a_n]$$

is a even continued fraction of $Q_{1/2}(p/q)$. Hence

$$g(Q_0(p/q)) \overset{3.17}{=} \frac{1}{2}n - 1 = \frac{1}{2}((n-1)-1) - 1 + 1 \overset{3.17}{=} g(Q_{1/2}(p/q)) + 1.$$

Interpreting these cases in rational numbers using

$$[0, 2, a_3, \ldots, a_n] = \frac{1}{2 + \underbrace{[a_3, \ldots, a_n]^{-1}}_{\in(-1,1)}}$$

we get the claim. ∎

4 Heegaard Floer invariants

Heegaard Floer theory was developed by Peter Ozsváth and Zoltán Szabó in the early 2000s using Heegaard splitting and Lagrangian Floer homology [OS04d]. They build some invariants for closed, oriented 3-manifolds equipped with a Spin^c-structure. Using this Heegaard Floer homologies Oszváth-Szabó [OS04c] and Jacob Rasmussen [Ras03] shortly afterwards independently defined a Floer invariant for null-homologous knots in oriented 3-manifolds, which was soon generalized for links.

This so called *knot Floer homology* takes the form of a bigraded finitely generated abelian group and contains information about several non-trivial geometric properties of the link (genus, slice genus, fibredness, effects of surgery, etc.). There is also a refinement for links called *link Floer homology* which admits a multigrading dependent on the number of link components. For an introduction see [Man16] or the overview [OS17] from Oszváth and Szabó themselves.

However, in this work we are primarily concerned about the probably simplest variant of knot Floer homology taking the form of a finite-dimensional bigraded vector space.

4.1 Knot Floer homology

Definition 4.1 (Bigraded vector space)**.** A **bigraded vector space** is a vector space V together with a decomposition

$$V = \bigoplus_{i,j\in\mathbb{Z}} V_{i,j}$$

where each $V_{i,j}$ is a vector space (implicitly this makes i the first grading).

A **morphism of bigraded vector spaces** is a linear map $f : V \to W$ between two bigraded vector spaces V and W such that

$$f(V_{i,j}) \subset W_{i,j}$$

for all $i, j \in \mathbb{Z}$. An **isomorphism of bigraded vector spaces** is a morphism of bigraded vector spaces which admits an inverse morphism of bigraded vector spaces.

A **relative morphism of bigraded vector spaces** is a linear map $f : V \to W$ between two bigraded vector spaces V and W such that there exist $a, b \in \mathbb{Z}$ such that

$$f(V_{i,j}) \subset W_{i+a,j+b}$$

for all $i, j \in \mathbb{Z}$. A **relative isomorphism of bigraded vector spaces** is a morphism of bigraded vector spaces which admits an inverse relative morphism of bigraded vector spaces. If we speak about bigraded vector spaces up to relative isomorphisms we call them **relatively bigraded vector spaces**.

Furthermore, if V is absolutely bigraded, we write

$$V_{,*} \coloneqq \bigoplus_{i\in\mathbb{Z}} \left(\bigoplus_{j\in\mathbb{Z}} V_{i,j} \right)$$
$$V_{*,} \coloneqq \bigoplus_{j\in\mathbb{Z}} \left(\bigoplus_{i\in\mathbb{Z}} V_{i,j} \right)$$

for the (uni)graded vector spaces we get from collapsing one of the two gradings.

Definition 4.2 (Euler characteristic)**.** For a graded vector space W we define its **Euler characteristic** as

$$\chi W \coloneqq \sum_{i\in\mathbb{Z}} (-1)^i \dim W_i \quad \in \mathbb{Z}.$$

For a bigraded vector space V we define the **graded Euler characterisic** as

$$\chi_{gr} V \coloneqq \sum_{i,j\in\mathbb{Z}} (-1)^i t^j \dim V_{i,j} \quad \in \mathbb{Z}[t^{\pm 1}].$$

Remark 4.3 ([OS04c], [Ras03])**.** The **knot Floer homology** $\widehat{\mathrm{HFK}}(L)$ of an oriented link L is a finite-dimensional bigraded vector space

$$\widehat{\mathrm{HFK}}(L) = \bigoplus_{h,A\in\mathbb{Z}} \widehat{\mathrm{HFK}}_h(L, A)$$

such that the map

$$L \quad \mapsto \quad \widehat{\mathrm{HFK}}(L)$$

is invariant under link equivalence. The first grading h is called the **homological** or **Maslov grading** and A the **Alexander grading**. Furthermore, we define a convenient third grading

$$\delta \coloneqq A - h$$

called the **δ-grading**.

Definition 4.4 (Grading shifts)**.** Let V be a bigraded vector space with respect to two gradings δ and A. Then for $m, n \in \mathbb{Z}$ denote with

$$\delta^m t^m V$$

the bigraded vector space which results from shifting the δ-grading of V by m and the A-grading by n. If V is missing one of the two gradings (or both) we think of V as trivially graded for the missing grading, i.e. all generators are supported in zero.

From now on V shall denote the bigraded vector space

$$V \coloneqq \delta^0 (t^{\frac{1}{2}} \mathbb{F}_2 \oplus t^{-\frac{1}{2}} \mathbb{F}_2)$$

i.e. the two-dimensional vector space supported in the single δ-grading 0 and the two consecutive Alexander gradings $\frac{1}{2}$ and $-\frac{1}{2}$.

In the year 1928 James Alexander introduced a polynomial link invariant [Ale28] and thereby presumably laid the foundation for modern knot theory. This Alexander polynomial takes the form of a multivariable polynomial over the integers where the number of variables equal the number of components of the given link. However, in this work we will mainly use the univariant Alexander polynomial for all links (which we get by equating multiple variables) and assume throughout, that the Alexander polynomials are symmetrized (which is possible by [TF54]). This means that the coefficient with index i is the same as the one with $-i$ for all $i \in \mathbb{N}$.
Interestingly, Knot Floer homology can be thought of as a categorification of the Alexander polynomial. In fact the latter can be computed from the former.

Proposition 4.5 ([OS04c])**.** *Let L be an oriented link. Then*

$$\Delta_L(t) \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\mu(L)-1} \doteq \chi_{gr}(\widehat{\mathrm{HFK}}(L))$$

where $\Delta_L(t)$ is the Alexander polynomial *of L. The symbol $\doteq$ means equality up to a multiplication by a unit of $\mathbb{Z}[t^{\pm 1}]$.*

However, knot Floer homology is a strictly stronger invariant, as the Alexander polynomial only gives us a lower bound for the genus of a link but knot Floer homology hands us the actual genus.
Using Remark 3.5, we can use a Theorem by Yi Ni to get a convenient formula for the genus of a link (proved for knots in [OS04b]).

Theorem 4.6 ([Ni06, Theorem 1.1])**.** *Let L be an oriented link. Then*

$$g(L) = \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L,\, A) \neq 0\} - \mu(L) + 1$$

where $\widehat{\mathrm{HFK}}_*(L,\, \cdot)$ is the Alexander graded vector space we get from collapsing the Maslov grading.

Proof. Let $A_{\max} = \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L, A) \neq 0\}$. From [Ni06, Theorem 1.1] we get

$$A_{\max} = \frac{\mu(L) - \chi(L)}{2}$$

Furthermore remark 3.5 implies

$$\begin{aligned} 2g(L) &= 2 - \chi(L) - \mu(L) \\ &= 2 - \chi(L) + \mu(L) - 2\mu(L) \\ &= \mu(L) - \chi(L) - 2\mu(L) + 2 \end{aligned}$$

which shows the claim

$$g(L) = A_{\max} - \mu(L) + 1.$$

□

Since the unknot is the unique knot with genus zero, the last theorem implies that if we restrict ourselves to knots, $\widehat{\mathrm{HFK}}$ detects the unknot [OS04b]. In fact knot Floer homology also detects the trefoil knots and the figure eight knot [Ghi08], as well as the Hopf link [Ni07] and all unlinks [Ni06], [Ni10].

Proposition 4.7. *[Ni10, Proposition 1.4] Suppose L is an oriented n-component link. If the dimension of its knot Floer homology $\widehat{\mathrm{HFK}}(L)$ is 2^{n-1}, then L is the n-component unlink.*

There are also very recent results concerning the torus links $T(2,4)$, $T(2,6)$, $T(3,3)$, the link $L7n1$ and a whole class of other torus links [BM20]. These detection results often used a property of links called fibredness.

Definition 4.8 (Fibredness)**.** An oriented link $L \subset S^3$ is called **fibred** , if $S^3 \setminus L$ fibres over the circle and L is the oriented boundary of the fibre. In other words, if there is a fibration sequence $F \to S^3 \setminus L \to S^1$ where F is a Seifert surface of L.

Remark 4.9. The unknot is fibred, but unlinks with mutable components are not. The only fibred knots with genus one are the two trefoils and the figure eight knot.

Given a fibred link its Alexander polynomial is necessarily monic. We see in the following that knot Floer homology gives again a stronger statement.

Definition 4.10 (Monic knot Floer homology)**.** We call the knot Floer homology of an oriented link **monic** if the topmost filtration level with respect to the Alexander grading is one-dimensional. In other words, $\widehat{\mathrm{HFK}}(L)$ of an oriented link L is monic if for

$$A_{\max} := \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L, A) \neq 0\}$$

holds that

$$\dim \widehat{\mathrm{HFK}}_*(L, A_{\max}) = 1.$$

Ozsvát and Szabó showed in [OS05] that the knot Floer homology of a knot must be monic and conjectured in [OS04a] that the converse is true for knots in S^3. Ghiggini showed this for genus one knots in [Ghi08] with which he showed the detection of the trefoil knots and the figure eight knot. Finally, Ni proved this conjecture in his PhD thesis and gave a more general result for links.

Theorem 4.11 ([Ni07, Corollary 1.2])**.** *Let L be an oriented non-split link. Then L is fibred if and only if $\widehat{\mathrm{HFK}}(L)$ is monic.*

In general, if we fix an Alexander degree the knot Floer homology of an oriented link can have generators in multiple Maslov degrees. However, it is reasonable to think about links where Alexander and Maslov grading have a linear dependence.

Definition 4.12 (δ-thin)**.** We call the knot Floer homology of an oriented link **δ-thin** if it is supported in only one δ-grading. An oriented link is called **δ-thin** if its knot Floer homology is so.

These δ-thin links give us a nice subset of all links, in so far that for them the Alexander polynomial and knot Floer homology encompass (almost) equal information. The only thing that gets lost is a grading shift of the δ-grading. This rests on the fact that when we depict the generators of a δ-thin knot Floer homology in a h-A-diagram, the generators lie on a single diagonal hence there can be no cancellations in the decategorification.

Lemma 4.13. *The knot Floer homology of an oriented δ-thin link can be computed (up to a δ-grading shift) from its Alexander polynomial. More precisely: Let*

$$\Delta_L(t) = \sum_{i=-n/2}^{n/2} a_i t^i \in \mathbb{Z}[t^{\pm 1/2}]$$

be the Alexander polynomial of L (with $a_{n/2} \neq 0$), then

$$\widehat{\mathrm{HFK}}(L) \;\cong\; \left(\bigoplus_{i=-n/2}^{n/2} \delta^0 t^i \mathbb{F}_2^{|a_i|} \right) \otimes V^{\otimes(\mu(L)-1)}.$$

Where the isomorphism is relative with respect to the δ-grading and absolute with respect to the Alexander grading.

Proof. This follows from proposition 4.5 if we remember that $\delta = A - h$ and use that δ is constant. □

This small lemma has some nice corollaries.

Corollary 4.14. *Let L be an oriented δ-thin link. Then*

$$g(L) = \frac{1}{2}(\deg \Delta_L(t) - \mu(L) + 1).$$

Proof. From lemma 4.13 it is clear that the topmost occurring Alexander grading of $\widehat{\mathrm{HFK}}(L)$ is given by equals $\frac{\deg \Delta_L(t) + (\mu(L)-1)}{2}$. Hence theorem 4.6 gives

$$\begin{aligned} g(L) &= \frac{\deg \Delta_L(t) + (\mu(L) - 1)}{2} - \mu(L) + 1 \\ &= \frac{1}{2}(\deg \Delta_L(t) - \mu(L) + 1). \end{aligned}$$

□

Corollary 4.15. *Let L be an oriented δ-thin link. Then L is fibred if and only if $\Delta_L(t)$ is monic.*

Proof. Follows directly from lemma 4.13. □

Proposition 4.16 ([OS03],[OS08]). *Alternating non-split oriented links are δ-thin.*

We can see that we need the non-split condition in the last proposition when we look at the unlink U_n for $n > 1$. Note that the Alexander polynomial is $\Delta_{U_n}(t) = 0$. From proposition 4.7 we know that $\dim \widehat{\mathrm{HFK}}(U_n) = 2^{n-1}$, therefore proposition 4.5 shows that there must be some cancellation when adding up the Euler characteristic. This can only happen for non-δ-thin links.

Remark 4.17 (δ-thin rational links)**.** We know that rational links are alternating (2.49) and that the only split rational link is the unlink (2.42), hence by proposition 4.16 all other oriented rational links are δ-thin. In particular, we know from corollary 2.47 that an oriented link $N(p/q)$ is δ-thin for all $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$.

Now we can compute the knot Floer homology (up to a δ-grading shift) of non-split alternating links even without any technical background about the construction of the homology. All we need is the Alexander polynomial which we could (for example) compute from a link diagram by using certain skein relations.

Example 4.18. The Alexander polynomial the unknot U is zero, hence lemma 4.13 says that

$$\widehat{\mathrm{HFK}}_*(U_1) \cong t^0\mathbb{F}_2^0$$

Of course, this shows that the unknot has genus zero, but it also shows that the unknot is fibred.

The oriented Hopf links $H_\pm$ have the Alexander polynomial

$$\Delta_{H_\pm}(t) = t^{-1/2} - t^{1/2}$$

hence

$$\widehat{\mathrm{HFK}}_*(U_1) \cong (t^{-1/2}\mathbb{F}_2 \oplus t^{1/2}\mathbb{F}_2) \otimes V = t^{-1}\mathbb{F}_2 \oplus t^0\mathbb{F}_2^2 \oplus t^1\mathbb{F}_2.$$

This shows that the Hopf links have genus zero and are fibred.

Another example would be the left-handed (or right-handed) trefoil K with

$$\Delta_K(t) = t^{-1} - 1 + t$$

and hence

$$\widehat{\mathrm{HFK}}_*(K) \cong t^{-1}\mathbb{F}_2 \oplus t^0\mathbb{F}_2 \oplus t^1\mathbb{F}_2.$$

This shows that K has genus one and is fibred, but we also see the loss of information caused by the relative δ-grading. The Alexander polynomial cannot distinguish the left-handed and right-handed trefoil knot cause of its mirror symmetry. Yet, knot Floer homology supports them in different δ-grading [OS04c] and can hence distinguish them. But mind that in this case (and generally for alternating knots) we could remedy this by looking at the signature of the given knot [OS03, Theorem 1.3].

4.2 A multicurves tangle invariant

In his PhD thesis from 2017 ([Zib17]) Claudius Zibrowius introduced a "local" version of link Floer homology, more precisely a Heegaard Floer invariant $\widehat{\mathrm{HFT}}$ for tangles T called the *tangle Floer homology* ([Zib17, Theorem 0.1]). Restricted to Conway tangles (4-ended tangles) this tangle Floer homology leads to an bordered sutured Heegaard Floer invariant called the *peculiar module* $\mathrm{CFT}^{\partial}(T)$ ([Zib17, Theorem 0.6]). These peculiar modules are combinatorially computable ([Zib20, Theorem 0.8]) and have a have a particularly nice gluing property ([Zib20, Theorem 0.4]). Furthermore, Claudius characterized this peculiar modules using collections of immersed loops (up to homotpy) in the (parametrized) four punctured sphere ([Zib20, Theorem 0.2]). This was shown using an equivalence between the category of peculiar modules and the compact part of the Fukaya category of the 4-punctured sphere S^2_4. The multicurve invariant $\mathrm{HFT}(T)$ then also satisfies a quite geometric gluing property recovering the link Floer homology ([Zib20, Corollary 0.7]). Further investigation of this immersed curve invariant, in particular concerning the mapping class group $\mathrm{Mod}(S^2_4)$, was done in [Zib19].

Definition 4.19 (Primitive elements)**.** Let G be a group. We call an element $g \in G$ **primitive** if there does not exist $h \in G$ and $n \in \mathbb{N}_{>2}$ such that

$$g = h^n.$$

We call a loop in S^2_4 **primitive**, if it defines a primitive element of $\pi_1(S^2_4)$.

The following definitions are according to [Zib19].

Definition 4.20 (Multicurve)**.** A **ζ-loop** or **immersed curve with local system** on the 4-punctured sphere S^2_4 is a pair (γ, X) where γ is an immersion of an oriented circle into S^2_4 resulting in a primitive curve and $X \in \mathrm{GL}_n(\mathbb{F}_2)$ for some integer $n \in \mathbb{N}_{>0}$. We consider γ up to homotopy, X up to matrix similarity and (γ, X) up to simultaneous orientation reversal of γ and inversion of the matrix X. Given an ζ-loop (γ, X), we call γ its underlying curve and X its local system.

A **multicurve** is an unordered set of ζ-loops on S^2_4 each of whose local system is the companion matrix of a polynomial in $\mathbb{F}_2[X]$, subject to the condition that each set of polynomials whose corresponding curves are homotopic to each other can be ordered such that one polynomial divides the next.

Remark 4.21 (The invariant HFT)**.** The Conway tangle invariant

$$T \to \mathrm{HFT}(T)$$

associates to every Conway tangle T a multicurve $\mathrm{HFT}(T)$ ([Zib19, Definition 1.12]). If T is oriented, $\mathrm{HFT}(T)$ is equipped with a relative bigrading.

Definition 4.22. Given a map $\gamma : S^1 \to S^2_4$, its **lift** along η is a map $\tilde{\gamma} : I \to \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that the diagram in figure 18 commutes.

$$\begin{array}{ccc} I & \xrightarrow{\tilde{\gamma}} & \mathbb{R}^2 \setminus \mathbb{Z}^2 \\ {\scriptstyle 0\sim 1}\downarrow & & \downarrow{\scriptstyle \eta} \\ S^1 & \xrightarrow{\gamma} & S^2_4 \end{array}$$

Figure 18: Lift of a curve γ

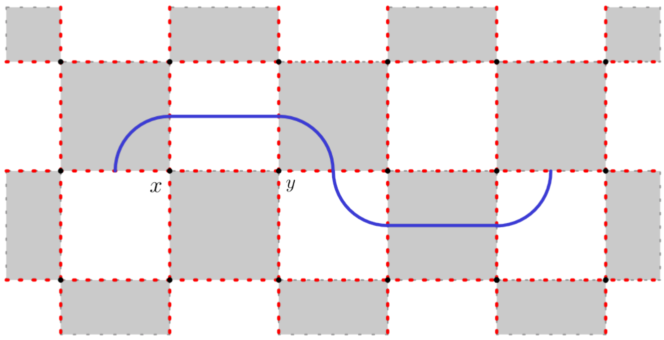


Figure 19: Lift of $\mathfrak{s}_1(0; x, y)$

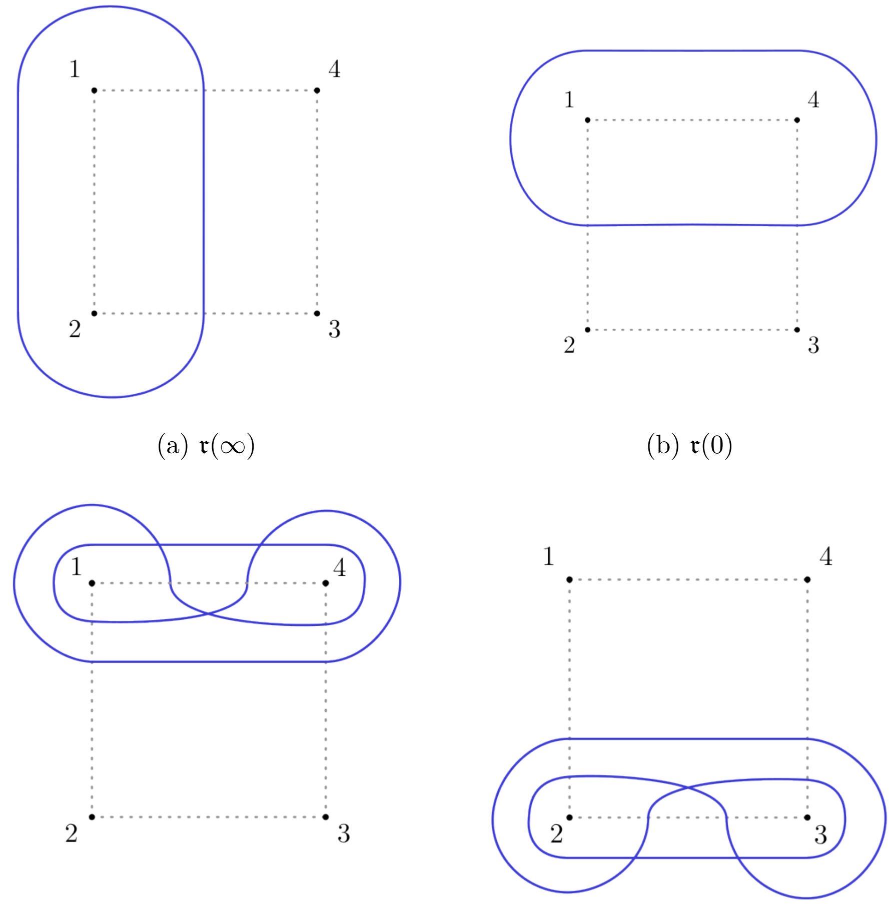
Definition 4.23 (Rational and special curves)**.** For a given slope $p/q \in \mathbb{Q}\mathrm{P}^1$, let $\mathfrak{r}(p/q)$ be an embedded, primitive loop in S^2_4 which under the covering map $\eta : \mathbb{R}^2 \setminus \mathbb{Z}^2 \to S^2_4$ lifts (up to homotopy) to a straight line of slope $p/q \in \mathbb{Q}\mathrm{P}^1$. We write $\mathfrak{r}_X(p/q)$ for this curve equipped with a local system $X \in \mathrm{GL}_m(\mathbb{F}_2)$ for some positive integer m and call it the **rational curve of slope** p/q **with local system** X. We will omit the 1-dimensional (trivial) local system X from the notation.

The second family of curves is constructed as follows. Suppose x and y are two distinct tangle ends which lie on a straight line of slope $p/q \in \mathbb{Q}\mathrm{P}^1$. The lattice points divide this line into intervals of equal length. For a fixed integer $n > 0$, let us mark every $(2n)$-th interval of the line. Then consider a small push-off of this line such that it intersects only the marked intervals and each of them exactly once. Finally, let $\mathfrak{s}_n(p/q; x, y)$ be an immersed, primitive loop in S^2_4 which under η lifts to this push-off (see figure 19). We call $\mathfrak{s}_n(p/q; x, y)$ the **special curve of slope** p/q **through the punctures** x **and** y. Note that for each slope p/q and fixed $n > 0$, there are exactly two choices for the pair of punctures (x, y).

If a primitive loop in S^2_4 is either rational or special we call it **linear with slope** p/q.

Theorem 4.24 ([Zib19, Theorem 0.5])**.** *For a Conway tangle T, the underlying curve of each component of* $\mathrm{HFT}(T)$ *is either rational or special. Moreover, if it is special, its local system is equal to an identity matrix.*

Remark 4.25 (About local systems)**.** The local systems of multicurves arise in the construction and classification of this invariant. However, they are of no importance for us, which is why we will ignore them entirely and assume that all curves have the trivial local system.



(a) $\mathfrak{r}(\infty)$ (b) $\mathfrak{r}(0)$

(c) $\mathfrak{s}_1(0; 1, 4)$ (d) $\mathfrak{s}_1(0; 2, 3)$

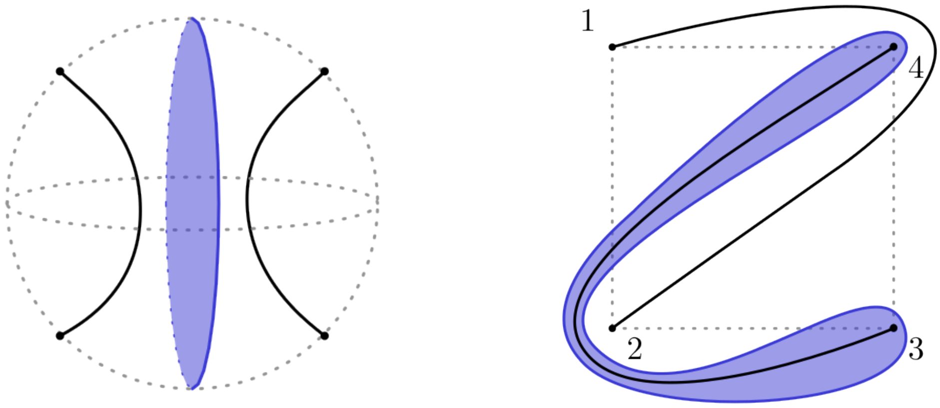
Figure 20: Examples of linear curves

There are also reasons for this. A curve with the identity matrix as local system can be interpreted as multiple copies of parallel curves with trivial local system. Therefore the last theorem 4.24 implies that, if at all, local systems are only important for rational curves. Yet, it is unknown wheter rational curves with non-trivial local system occur in the multicurve invariant of Conway tangles. At least they do not appear for the tangles we mainly look at (Theorem 4.28).

Besides the last theorem, we know that special components of a multicurve $\mathrm{HFT}(T)$ admit a (so-called) conjugation symmetry.

Theorem 4.26 ([LMZ21, Theorem 3.9])**.** *Let $i, j, k, l \in \mathbb{N}$ such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Moreover, let $p/q \in \mathbb{Q}\mathrm{P}^1$ and $n \in \mathbb{N}$. Then, for any Conway tangle T, the number of components of the form $\mathfrak{s}_n(p/q; i, j)$ and $\mathfrak{s}_n(p/q; k, l)$ in $\mathrm{HFT}(T)$ agree.*

The invariant HFT commutes with the action of the modular class group.



(a) A compressing disk for Q_∞ (resp. Q_0)

(b) A compressing disk for $Q_{1/2}$

Figure 21: Compressing disks of rational tangles

Theorem 4.27 ([Zib19, Theorem 2.1])**.** *Let T be a Conway tangle and $\tau \in \mathrm{Mod}(S^2_4(T))$. Then*

$$\mathrm{HFT}(\tau T) = \tau(\mathrm{HFT}(T)).$$

Finally, invariant HFT detects rational tangles.

Theorem 4.28 ([Zib20, Theorem 6.2])**.** *A Conway tangle T is rational if and only if* $\mathrm{HFT}(T)$ *consists of single rational component carrying the unique one-dimensional local system. If the tangle is rational, the rational component has the same slope as the tangle.*

Remark 4.29 (HFT of rational tangles)**.** We wamt to justify the last theorem 4.28 a little bit. Given an rational tangle Q we know by Conway's algorithm that Q can be written as

$$Q \coloneqq \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

for some continued fraction $[a_1, \ldots, a_n]$. Theorem 4.27 implies that

$$\mathrm{HFT}(Q) \coloneqq \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\,\mathrm{HFT}(Q_\infty), & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\,\mathrm{HFT}(Q_0), & \text{if } n \text{ is odd.} \end{cases}.$$

Given that $\mathrm{HFT}(Q_\infty)$ and $\mathrm{HFT}(Q_0)$ look like $\mathfrak{r}(\infty)$ and $\mathfrak{r}(0)$ in figure 20, we're able to determine $\mathrm{HFT}(Q)$: The invariant $\mathrm{HFT}(T)$ for $T \in \{Q_\infty, Q_0\}$ is precisely the border ∂D of a compressing disk D of $B_T \setminus \operatorname{im} T$ (Figure 21a). Remember that rational tangles are split (Corollary 2.42) and such a compressing disk therefore exists (Definition 2.14). Let $\tau \in \mathrm{Mod}(S^2_4)$ such that $Q = \tau T$. We get that

$$\mathrm{HFT}(Q) = \mathrm{HFT}(\tau T) \overset{4.27}{=} \tau\,\mathrm{HFT}(T) = \tau\partial D.$$

Furthermore, because τ can be interpreted as an automorphisms of B_T (Fact 2.17), τD is a compressing disk of $B_Q \setminus \operatorname{im} Q$. More practically, we can take a crossing-less diagram of Q, which exists by lemma 2.31, and take the border of a small closed neighbourhood of any of the two open components to get $\mathrm{HFT}(Q)$. For $Q = Q_{1/2}$ this is depicted in figure 21b.

The last detection result was expanded to split tangles.

Theorem 4.30 ([LMZ21, Theorem 4.1])**.** *A Conway tangle T is split if and only if $\mathrm{HFT}(T)$ only contains rational components that all have the same slope. If T is split, the local systems on those components are trivial.*

After this short overview, we want to prove some results concerning the geometry of rational curves.

Lemma 4.31. *Let L be a straight line in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ with slope $p/q \in \mathbb{QP}^1$. If we start at an arbitrary $x \in L$ and follow L, the first $y \in L$ with*

$$\eta(x) = \eta(y)$$

is given by

$$y = x + \begin{pmatrix} 2q \\ 2p \end{pmatrix} \quad \text{or} \quad y = x - \begin{pmatrix} 2q \\ 2p \end{pmatrix}$$

depending on the direction we go on L.

Proof. Remember the definition of η in section 2.2. In formulas this means that two points $x, y \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ have the same image $\eta(x) = \eta(y)$ if and only if

$$x - y \in (2\mathbb{Z})^2 \quad \text{or} \quad x + y \in (2\mathbb{Z})^2.$$

We examine the case where we go in positive direction. As L is a straight line with slope p/q, we can reformulate the question: We want the smallest $\lambda \in \mathbb{R}_{>0}$ such that

$$x - (x + \lambda \begin{pmatrix} q \\ p \end{pmatrix}) \in (2\mathbb{Z})^2 \quad \text{or} \quad x + (x + \lambda \begin{pmatrix} q \\ p \end{pmatrix}) \in (2\mathbb{Z})^2$$

respectively

$$-\lambda \begin{pmatrix} q \\ p \end{pmatrix} \in (2\mathbb{Z})^2 \quad \text{or} \quad 2x + \lambda \begin{pmatrix} q \\ p \end{pmatrix} \in (2\mathbb{Z})^2.$$

As p and q cannot both be even, we get $\lambda = 2$ as the smallest solution for the first condition. For the second condition we have to mind that L lies in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ and hence no point on L can have integer coordinates. If there would be a λ such that

$$2x + \lambda \begin{pmatrix} q \\ p \end{pmatrix} \in (2\mathbb{Z})^2,$$

we would have

$$x + \frac{\lambda}{2}\begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{Z}^2,$$

i.e. a point on L with integer coordinates. Hence $\lambda = 2$ is the smallest solution of both conditions. Analogously prove the case for the negative direction. $\square$

Lemma 4.32. *Let $\gamma \coloneqq \mathfrak{r}(p/q)$ be a rational curve of slope $p/q \in \mathbb{QP}^1$ on S^2_4. For any lift $\tilde{\gamma}$ of γ holds that*

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) + \begin{pmatrix} 2q \\ 2p \end{pmatrix} \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0) - \begin{pmatrix} 2q \\ 2p \end{pmatrix}.$$

Proof. By definition γ is homotopic to a loop γ' which lifts along η to a straight line of slope p/q. Via the identification

$$\frac{I}{0 \sim 1} \cong S^1$$

we interpret this homotopy as map

$$H : I \times I \to S^2_4$$

with

$$H(0, \cdot) = \gamma$$
$$H(1, \cdot) = \gamma'$$
$$\alpha \coloneqq H(\cdot, 0) = H(\cdot, 1).$$

Let $\tilde{\gamma}$ be a lift of γ along η. As η is a covering space, we can lift H uniquely to a continuous map

$$\tilde{H} : I \times I \to \mathbb{R}^2 \setminus \mathbb{Z}^2$$

such that $\tilde{H}(0, \cdot) = \tilde{\gamma}$. By commutativity of diagrams $\tilde{\gamma}' \coloneqq \tilde{H}(1, \cdot)$ is a lift of γ' and hence a straight line of slope p/q. From lemma 4.31 we get that

$$\tilde{\gamma}'(1) - \tilde{\gamma}'(0) = \lambda \begin{pmatrix} 2q \\ 2p \end{pmatrix} \tag{$*$}$$

where $\lambda \in \{-1, 1\}$. Note that the translation

$$t : \mathbb{R}^2 \setminus \mathbb{Z}^2 \to \mathbb{R}^2 \setminus \mathbb{Z}^2$$
$$x \mapsto x + \lambda \begin{pmatrix} 2q \\ 2p \end{pmatrix}$$

is a deck transformation of η. We know that $\tilde{H}(\cdot, 0)$ and $\tilde{H}(\cdot, 1)$ are two lifts of the curve α. The curves $t \circ \tilde{H}(\cdot, 0)$ and $\tilde{H}(\cdot, 1)$ are therefore also two lifts along $\eta = t \circ \eta$ of α with

$$t \circ \tilde{H}(0, 0) = t(\tilde{\gamma}(0)) \overset{(*)}{=} \tilde{\gamma}(1) = \tilde{H}(0, 1).$$

The unique path lifting property then shows that $t \circ \tilde{H}(\cdot, 0) = \tilde{H}(\cdot, 1)$. In particular

$$\tilde{H}(1,0) - \tilde{H}(0,0) = t(\tilde{H}(1,0)) - t(\tilde{H}(0,0)) = \tilde{H}(1,1) - \tilde{H}(0,1). \tag{$**$}$$

Hence

$$\begin{aligned}
\tilde{\gamma}(1) - \tilde{\gamma}(0) &= \tilde{\gamma}(1) + (-\tilde{\gamma}'(1) + \tilde{\gamma}'(1)) + (-\tilde{\gamma}'(0) + \tilde{\gamma}'(0)) - \tilde{\gamma}(0) \\
&= \tilde{H}(0,1) - \tilde{H}(1,1) + (\tilde{\gamma}'(1) - \tilde{\gamma}'(0)) + \tilde{H}(1,0) - \tilde{H}(0,0) \\
&\overset{(**)}{=} \tilde{\gamma}'(1) - \tilde{\gamma}'(0) \\
&\overset{(*)}{=} \lambda \begin{pmatrix} 2q \\ 2p \end{pmatrix}.
\end{aligned}$$

□

Remark 4.33 (On minmal intersections)**.** Let $p/q \in \mathbb{Q}\mathrm{P}^1$ and let $\gamma := \mathfrak{r}(p/q)$ lie on $S^2_4 := S^2_4(T)$ for an oriented tangle T. As always, we assume that the parametrization of S^2_4 lifts along η to the standard unit grid of $\mathbb{R}^2 \setminus \mathbb{Z}^2$. We want to determine the minimal number of intersections of γ with the sites a, b, c, d which depends on the chosen representative.

Assume that γ intersects the site a minimally. Let $\tilde{\gamma}$ be a lift of γ along η, then lemma 4.32 gives us that

$$\tilde{\gamma}(1) - \tilde{\gamma}(0) = \lambda \begin{pmatrix} 2q \\ 2p \end{pmatrix} \tag{$*$}$$

for $\lambda \in \{-1, 1\}$. We compute the number $N \in \mathbb{N}$ of intersections between $\operatorname{im} \tilde{\gamma}|_{[0,1)}$ and $\eta^{-1}(a)$ which are in bijection with the intersections between $\operatorname{im} \gamma$ and a. Because of $(*)$ we know that

$$|q| \leq N$$

as otherwise $\tilde{\gamma}$ would not be continuous (basically the intermediate value theorem). Furthermore, γ is homotopic to a loop γ' which lifts along η to a straight line $\tilde{\gamma}'$ of slope p/q. This straight line satisfies that

$$|\operatorname{im} \tilde{\gamma}'|_{[0,1)} \cap \eta^{-1}(a)| = |q|,$$

hence

$$N \leq |q|$$

as γ intersects the site a minimally by assumption. This gives us that

$$|\operatorname{im} \gamma \cap a| = |\operatorname{im} \tilde{\gamma}|_{[0,1)} \cap \eta^{-1}(a)| = N = |q|.$$

With the analogous argument for b, c and d we get

$$\begin{aligned}
|\operatorname{im} \gamma \cap b| &= |p|, \\
|\operatorname{im} \gamma \cap c| &= |q|, \\
|\operatorname{im} \gamma \cap d| &= |p|.
\end{aligned}$$

Because γ' satisfies all these equalities at the same time, we know that $\mathfrak{r}(p/q)$ intersects the parametrization minimally if it lifts to a straight line of slope p/q.

Lemma 4.34. *Let $p/q \in \mathbb{Q}\mathrm{P}^1$ and let all curves lie on some four-punctured sphere S^2_4. If*

$$|\mathfrak{r}(p/q) \cap \mathfrak{r}(0)| = 2|p|$$

then $\mathfrak{r}(p/q)$ and $\mathfrak{r}(0)$ intersect minimally. Furthermore, if

$$|\mathfrak{r}(p/q) \cap \mathfrak{s}_1(0; x, y)| = 4|p|$$

then $\mathfrak{r}(p/q)$ and $\mathfrak{s}_1(0; x, y)$ for $(x, y) \in \{(1, 4), (2, 3)\}$ intersect minimally.

Proof. Fix a representative of $\gamma_0 := \mathfrak{r}(0)$ that lifts along η to a straight line of slope zero. Furthermore we can homotope γ_0 such that it lies in an arbitrarily small neighbourhood of the site d (see figure 29d). If we homotope γ such that it intersects the parametrization and γ_0 minimally in $N \in \mathbb{N}$ points, the picked representative of γ_0 and remark 4.33 imply that

$$2|p| = 2|\mathrm{im}\,\gamma \cap d| \leq N.$$

Furthermore, a representative of γ that lifts to a straight line of slope p/q satisfies

$$N = 2|\mathrm{im}\,\gamma \cap d|$$

hence

$$N = 2|p|.$$

In this proof we also saw that γ and γ_0 can be homotoped such that they intersect minimally and both lift to straight lines.

For $\gamma_s := \mathfrak{s}_1(0; x, y)$ we have to mind that γ_s is homotopic to the curve in figure 24. We see that γ_s can be homotoped such that it lies in an arbitrarily small neighbourhood of the site d. If we homotope γ such that it intersects the parametrization and γ_0 minimally in $N \in \mathbb{N}$ points, we get for $s \in \{b, d\}$ that

$$4|p| = 4|\mathrm{im}\,\gamma \cap s| \leq N.$$

And again, a representative of γ that lifts to a straight line of slope p/q satisfies

$$N = 4|\mathrm{im}\,\gamma \cap s|$$

hence

$$N = 4|p|.$$

□

The invariant HFT is equipped with a relative bigrading [Zib19, Definition 1.10]. This relative bigrading takes the form of a relative bigrading on the intersection of $\mathrm{HFT}(T)$ with the parametrization arcs a, b, c, d. However, to make this work we *need to assume that the multicurve intersects the parametrization minimally.* We will assume this throughout this work for every linear curve on a parametrized four-punctured sphere. It is sufficient for our purposes to know how the relative bigrading behaves along single linear curves.

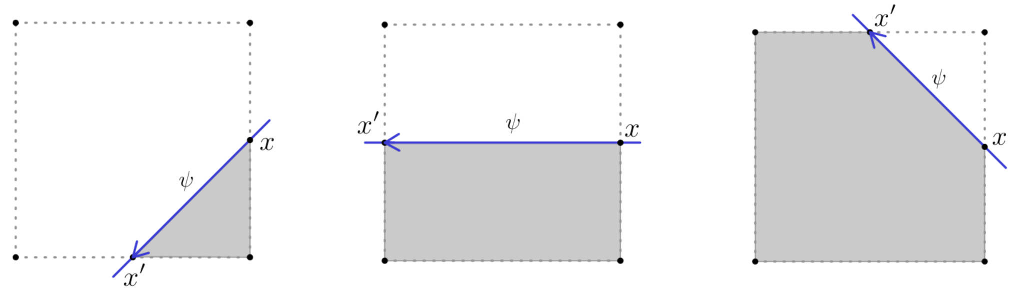


Figure 22: Basic regions to compute the bigrading of a curve

Definition 4.35 (Ordered matching)**.** A **matching** P is a partition $\{\{i_1, o_1\}, \{i_2, o_2\}\}$ of $\{1, 2, 3, 4\}$ into pairs. An **ordered matching** is a matching in which the pairs are ordered. A oriented Conway tangle T gives rise to a matching P_T as follows: The pairs consist of the endpoints of open components of T. Given an orientation of the two open components of T, we order each pair of points such that the inwardly pointing end comes first, the outwardly pointing end second.

Definition 4.36. Let $P = \{(i_1,\, o_1),\, (i_2,\, o_2)\}$ be an ordered matching. Then we define the $\mathbb{Z}$-module

$$\mathfrak{A}_P := \frac{\mathbb{Z}^4}{(e_{i_1} + e_{o_1}, e_{i_2} + e_{o_2})}$$

where e_j is the j^{th} unit vector in $\mathbb{Z}^4$, as well as the $\mathbb{Z}$-linear map

$$\begin{aligned} \mathrm{u}_P :\ \mathfrak{A}_P &\to \tfrac{1}{2}\mathbb{Z} \\ (a_1, a_2, a_3, a_4) &\mapsto \frac{1}{2}\sum_{k=1}^{4} \varepsilon_k a_k \end{aligned}$$

where $\varepsilon_{i_1} = \varepsilon_{i_2} = 1$ and $\varepsilon_{o_1} = \varepsilon_{o_2} = -1$.

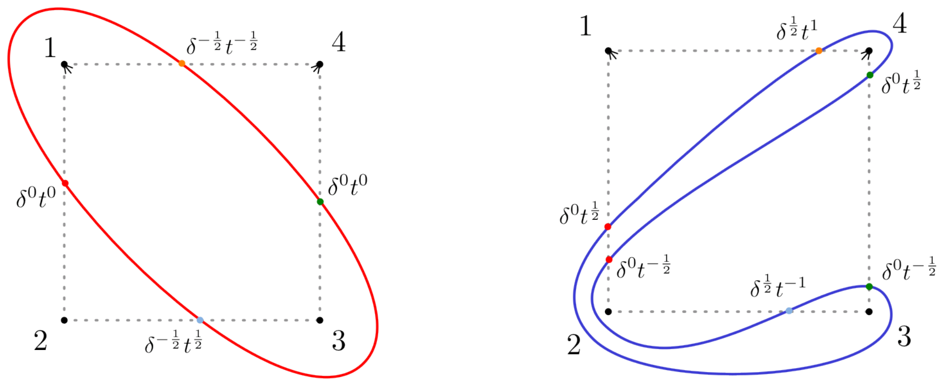
The next remark is based on [LMZ21, Section 3.5].

Remark 4.37 (Bigrading of HFT)**.** Let T be an oriented Conway tangle and $S^2_4 := S^2_4(T)$. Let γ be a linear on S^2_4 and $\mathcal{G}(\gamma)$ be the the set of intersection points between the sites a, b, c, d and γ.

A δ-grading on γ is a function

$$\delta : \mathcal{G}(\gamma) \to \frac{1}{2}\mathbb{Z}$$

which admits the following conditions: Suppose $x, x' \in \mathcal{G}(\gamma)$ are two intersection points such that there is a path ψ on γ which connects x to x' without meeting any parametrizing arc, except at the endpoints. We distinguish three cases, which are illustrated in



(a) HFT(Q_{-1}) (b) HFT($Q_{1/2}$)

Figure 23: Bigraded rational curves

figure 22. The path can turn left (a), it can go straight across (b), or it can turn right (c). Then

$$\delta(x') - \delta(x) = \begin{cases} \frac{1}{2}, & \text{if the path } \psi \text{ turns left,} \\ 0, & \text{if the path } \psi \text{ goes straight across,} \\ -\frac{1}{2}, & \text{if the path } \psi \text{ turns right.} \end{cases}$$

By definition 4.35 the tangle T gives rise an ordered matching $P_T = \{(i_1, o_1), (i_2, o_2)\}$. Having that, we define the multivariate Alexander grading as function

$$\hat{A} : \mathcal{G}(\gamma) \to \mathfrak{A}_P$$

satisfying the following compatibility condition: If ψ is a path from x to x' as above, then

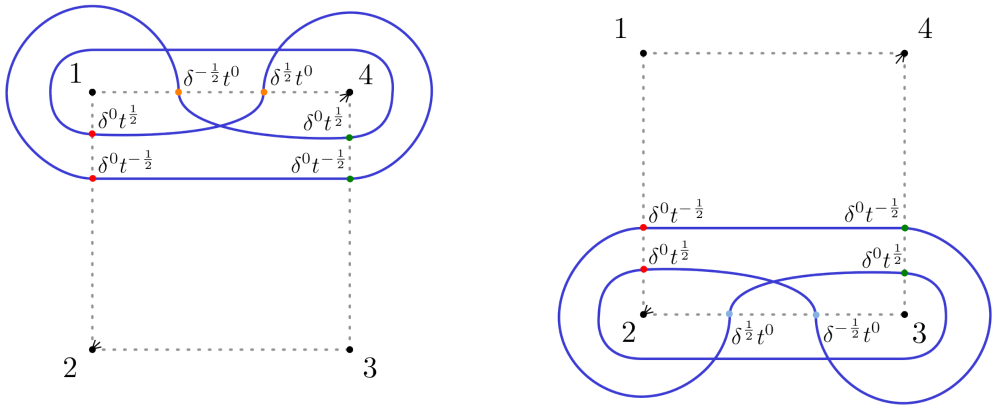
$$\hat{A}(x') - \hat{A}(x) = (a_1, a_2, a_3, a_4) \in \mathfrak{A}_P, \text{ where } a_j := \begin{cases} -1, & \text{if } j \text{ lies to the left of } \psi, \\ 0, & \text{if } j \text{ lies to the right of } \psi. \end{cases}$$

The univariate Alexander grading is then defined as

$$A := \mathrm{u}_P \circ \hat{A} : \mathcal{G}(\gamma) \to \frac{1}{2}\mathbb{Z}.$$

In this way we get a *unique relative bigrading* on linear curves. In the same manner HFT(T) for a oriented Conway tangle T is equipped with relative bigrading (possibly over multible curves). Most of the time we will fix an *absolute bigrading* on linear curves, by simply setting the bigrading of a single point. These absolute bigradings are then clearly equal up to a shift.

Example 4.38. In figure 23 we can see the absolutely bigraded multicurve invariants HFT(Q_{-1}) and HFT($Q_{1/2}$) for the tangles Q_{-1} and $Q_{1/2}$ oriented as depicted (the small arrows). In figure 24 we see the absolutely bigraded special curves $\mathfrak{s}_1(0; 1, 4)$ and $\mathfrak{s}_1(0; 2, 3)$ for the depicted tangle end directions.



(a) $\mathfrak{s}_1(0;1,4)$ (b) $\mathfrak{s}_1(0;2,3)$

Figure 24: Bigraded special curves

Remark 4.39. In [KWZ21, Definition/Lemma 4.15] the authors shows that for a linear curve γ of slope $s \in \mathbb{Q}\mathrm{P}^1$ holds the following: Unless $s = \infty$, all intersection points of γ with the sites a, c have the same δ-grading $\delta_|(\gamma)$ and unless $s = 0$, all intersection points of γ with the sites b, d have the same δ-grading $\delta_-(\gamma)$. Moreover,

$$\delta_| = \begin{cases} \delta_- - \frac{1}{2} & \text{if } 0 < s < \infty, \\ \delta_- + \frac{1}{2} & \text{if } \infty < s < \infty. \end{cases}$$

Definition 4.40 (Grading shifts)**.** Given an absolutely bigraded linear curve γ. Then let us denote by $\delta^m t^A \gamma$ the bigraded linear curve with local system obtained from γ by shifting the δ-grading by $m \in \frac{1}{2}\mathbb{Z}$ and the Alexander grading by $A \in \frac{1}{2}\mathbb{Z}$.

We know from theorem 4.27 that the invariant HFT commutes with the mapping class group of S^2_4. Even more interesting, we can make statements about the behaviour of a fixed absolute bigrading under such a transformation. The following explanations are lengthy, but all the more important for later results.

Definition 4.41. Let T be an oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$. Then for $s \subset \{a, b, c, d\}$ let $X_s(\gamma)$ denote the set of intersection points between γ and the site s. We call $x \in X_s(\gamma)$ a **generator on the site** s.

Remark 4.42. Let T be a oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$ be absolutely bigraded. We want a relation between γ and $\tau\gamma$ for $\tau \in \{\tau_1, \tau_2\}$. We get this by studying the bimodule depicted in [Zib19, Figure 11 (b)] which says how the generators of γ behave under τ^ε with $\varepsilon \in \{-1, +1\}$. Let λ be -1 if the tangle end 3 of T is pointing outwards and 1 if it points inwards. There are the following cases:

- If $T(\infty)$ exists, $X_a(\gamma) = X_a(\tau_1^\varepsilon\gamma)$ as well as $X_c(\gamma) = X_c(\tau_1^\varepsilon\gamma)$ without grading shifts; but τ_1^ε creates for every generator $x \in X_c(\gamma)$ two *provisional* generators

$x_1 \in X_b(\tau_1^\varepsilon\gamma)$ and $x_2 \in X_d(\tau_1^\varepsilon\gamma)$, which inherit the bigrading of x plus a bigrading shift. Both get a δ-grading shift by $-1/2$, which is not important for us. The Alexander grading behaves as follows:

$$A(x_1) = A(x) + \frac{\varepsilon\lambda}{2} \quad \text{and} \quad A(x_2) = A(x) - \frac{\varepsilon\lambda}{2}$$

- If $T(0)$ exists, $X_b(\gamma) = X_d(\tau_2^\varepsilon\gamma)$ as well as $X_b(\gamma) = X_d(\tau_2^\varepsilon\gamma)$ without grading shifts; but τ_2^ε creates for every generator $x \in X_b(\gamma)$ two *provisional* generators $x_1 \in X_a(\tau_2^\varepsilon\gamma)$ and $x_2 \in X_c(\tau_2^\varepsilon\gamma)$, which inherit the bigrading of x plus a bigrading shift. Both get a δ-grading shift by $-1/2$, which is not important for us. The Alexander grading behaves as follows:

$$A(x_1) = A(x) + \frac{\varepsilon\lambda}{2} \quad \text{and} \quad A(x_2) = A(x) - \frac{\varepsilon\lambda}{2}$$

- If $T(\infty)$ does not exist $X_a(\gamma) = X_a(\tau_1^\varepsilon\gamma)$ without δ-grading shift and a Alexander grading shift of $-\frac{\varepsilon\lambda}{2}$ as well as $X_c(\gamma) = X_c(\tau_1^\varepsilon\gamma)$ without δ-grading shift and a Alexander grading shift of $\frac{\varepsilon\lambda}{2}$. Furthermore all existing generators on the sites b and d get an Alexander grading shift of $-\frac{\varepsilon\lambda}{2}$. Then, τ_1^ε creates for every generator $x \in X_c(\gamma)$ two *provisional* generators $x_1 \in X_b(\tau_1^\varepsilon\gamma)$ and $x_2 \in X_d(\tau_1^\varepsilon\gamma)$, which inherit the bigrading of x without Alexander grading shift. Both get a δ-grading shift by $-1/2$, which is not important for us.

- If $T(0)$ does not exist $X_d(\gamma) = X_d(\tau_2^\varepsilon\gamma)$ without δ-grading shift and a Alexander grading shift of $-\frac{\varepsilon\lambda}{2}$ as well as $X_b(\gamma) = X_b(\tau_2^\varepsilon\gamma)$ without δ-grading shift and a Alexander grading shift of $\frac{\varepsilon\lambda}{2}$. Furthermore all existing generators on the sites b and d get an Alexander grading shift of $-\frac{\varepsilon\lambda}{2}$. Then, τ_1^ε creates for every generator $x \in X_b(\gamma)$ two *provisional* generators $x_1 \in X_a(\tau_2^\varepsilon\gamma)$ and $x_2 \in X_c(\tau_2^\varepsilon\gamma)$, which inherit the bigrading of x without Alexander grading shift. Both get a δ-grading shift by $-1/2$, which is not important for us.

We call these new generators *provisional*, because each one of them could cancel itself with another generator on the same site. However these cancellations obey certain rules. Cancellations can only happen...

- ...between provisional and already existing generators, i.e. two provisional generators cannot cancel themselves.

- ...between generators in the same bigrading.

After these possible cancellations all remaining provisional generators lose the status of provisional and are in fact the generators of $\tau^\varepsilon\gamma$.

4.3 Lagrangian intersection Floer theory

In the next section we will see that there is a need to compute something called the *Lagrangian intersection Floer homology* of two immersed curves in S^2_4. However, we approach this computation in a practical way and mainly explain a simplified case and state some properties of the homology.

From [Aur13, Motivation 1.1]: *"Lagrangian intersection Floer homology was introduced by Andreas Floer in the late 1980s in order to study intersection properties of compact Lagrangian submanifolds in symplectic manifolds."*

In our case the symplectic manifold is the four-punctured sphere and the role of compact Lagrangian submanifolds is taken by unobstructed [Abo08, Definition 2.1] curves occurring in $\mathrm{HFT}(T)$ for a Conway tangle T. It is described in [Abo08] that in this two-dimensional case the resulting homology groups are combinatorial in nature and that the Floer homology of two minimally intersecting curves is (except for exceptions) generated by their intersection points.

We think about the Lagrangian Floer homology in a manner like in [LMZ21] and [KWZ21]. The following is the only situation which will occur to us: Let γ and γ' be two *non-homotopic* linear curves on S^2_4 which intersect transversely and minimally such that they do not cobound immersed annuli. The **Lagrangian Floer homology** $\mathrm{HF}(\gamma, \gamma')$ is the vector space generated by the intersection points between the two curves. Mind that two linear curves are non-homotopic if they have different slope.

If two linear curves on S^2_4 are absolutely bigraded we can define an absolute bigrading on the Lagrangian Floer homology. This can be read in [LMZ21, Section 3.6] for the Alexander grading and [KWZ21, Section 4] for the δ-grading. Generally, one can compute the bigrading using *connecting domains* in the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$ which have a convenient additive behavior. Here, we restrict ourselves to mention what is needed in in this work (with forumlas inspired by [Var21]).

Remark 4.43 (Bigrading of HF)**.** Let T be an oriented Conway tangle and $S^2_4 := S^2_4(T)$ and let $P := \{a, b, c, d\}$ be the parametrization sites. Furthermore, let γ and γ' be two absolutely bigraded *non-homotopic* linear curves on S^2_4 which intersect transversely and minimally. Then each generator $z \in \mathrm{HF}(\gamma, \gamma')$ comes from an intersection $x \in \operatorname{im}\gamma \cap \operatorname{im}\gamma'$ which up to homotopy looks like one of the cases from figure 25 for intersections $x \in \operatorname{im}\gamma \cap P$ and $y \in \operatorname{im}\gamma' \cap P$. The shaded area φ is called a *basic connecting domain* (ignore additional curve segments in φ), which gives us a colouring (red and blue) of γ and γ' with respect to φ. Let $P(\varphi) \subset \{1, 2, 3, 4\}$ be the set punctures adjacent to φ. For the puncture $i \in \{1, 2, 3, 4\}$ let

$$\varepsilon_i := \begin{cases} 1 & \text{if } i \text{ points inwardly,} \\ -1 & \text{if } i \text{ points outwardly,} \end{cases}$$

and let

$$\nu := \begin{cases} 1 & \text{if } \gamma \text{ is red with respect to } \varphi, \\ -1 & \text{if } \gamma' \text{ is red with respect to } \varphi. \end{cases}$$

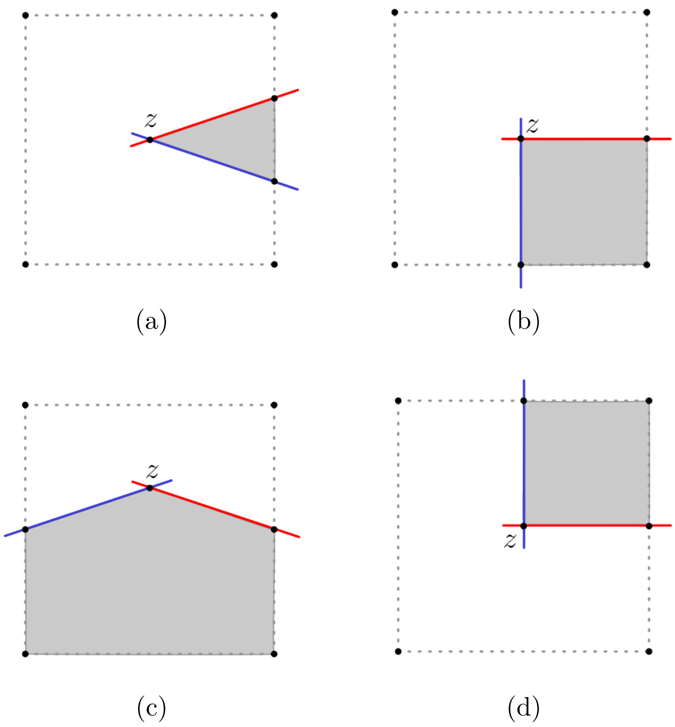


Figure 25: Basic connecting domains

The bigrading of z is then given by:

$$\delta(z) = \delta(y) - \delta(x) + \frac{1-\nu}{2} + \nu \sum_{p \in P(\varphi)} \frac{1}{2}$$
$$A(z) = A(y) - A(x) + \nu \sum_{p \in P(\varphi)} \frac{\varepsilon_p}{2}$$

Example 4.44. We examine the tangle $Q_{1/2}$ with ordered matching $\{(2,1),(3,4)\}$. Then $\mathrm{HFT}(Q_{1/2})$ and $\mathrm{HFT}(Q_{-1})$ from example 4.38 are two absolutely bigraded linear curves on $S^2_4(Q_{1/2})$. The slopes differ, hence the rational curves are non-homotopic. Furthermore, lemma 4.51 with

$$2|-1 \cdot 2 - 1 \cdot 1| = 6$$

shows that the loops have a minimal intersection number of six. Thus, we can compute the Lagrangian Floer homology

$$\mathrm{HF}(\mathrm{HFT}(Q_{-1}), \mathrm{HFT}(Q_{1/2}))$$

using the diagram figure 26. The basic connecting domains are coloured green if $\nu = 1$ and red if $\nu = -1$.

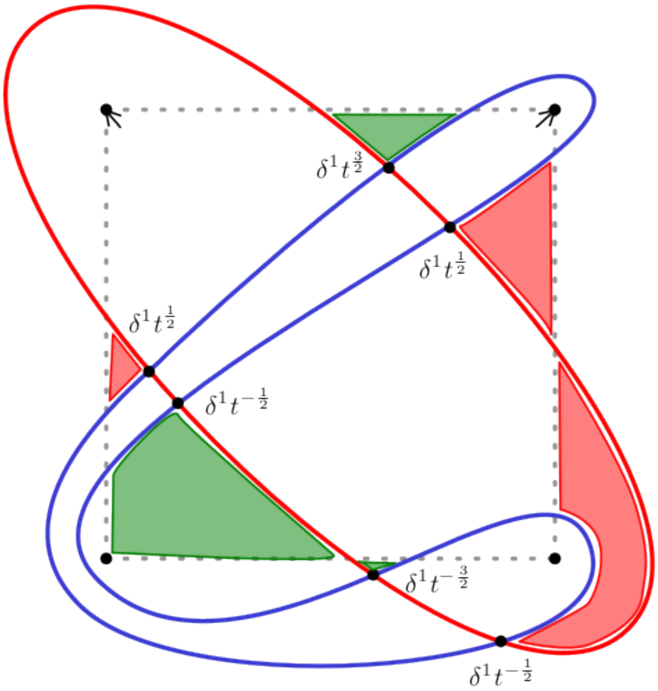


Figure 26: Pairing of $\mathrm{HFT}(Q_{-1})$ and $\mathrm{HFT}(Q_{1/2})$

If we find ourselves faced with an absolutely bigraded *multicurves*, we use the following fact:

Fact 4.45. *Let S^2_4 be the parametrized four-punctured sphere associated to an oriented Conway tangle. Let $\Gamma \coloneqq \{\gamma_i \mid i \in \{1, \ldots, n\}\}$ for $n \in \mathbb{N}$ be an absolutely bigraded multicurve and α be an absolutely bigraded ζ-loop on S^2_4. Then*

$$\mathrm{HF}(\alpha, \Gamma) = \bigoplus_{i=1}^{n} \mathrm{HF}(\alpha, \gamma_i)$$

as absolutely bigraded vector spaces.

Remark 4.46. Let S^2_4 be the parametrized four-punctured sphere associated to an oriented Conway tangle and Γ_1, Γ_2 be absolutely bigraded multicurves on S^2_4. Then it holds for $m, n \in \mathbb{N}$ that

$$\mathrm{HF}(t^m\Gamma_1,\, t^n\Gamma_2) = t^{n-m}\,\mathrm{HF}(\Gamma_1,\, \Gamma_2)$$

as absolutely bigraded vector spaces.

The next conjecture is a small unsightlinesss. Yet, it should be correct, even if there is yet no reference for it.

Conjecture 4.47. *Let S^2_4 be the parametrized four-punctured sphere associated to an oriented Conway tangle and* Γ_1, Γ_2 *be absolutely bigraded multicurves on S^2_4. Then it holds for $\tau \in \mathrm{Mod}(S^2_4)$ that*

$$\mathrm{HF}(\Gamma_1, \Gamma_2) = \mathrm{HF}(\tau\Gamma_1, \tau\Gamma_2)$$

as absolutely bigraded vector spaces.

Remark 4.48. We need this conjecture for the proof of proposition 4.70, which is then needed in the concluding propositions 6.8 and 6.9. But mind that the weaker statement for rational curves instead of multicurves would suffice. Proposition 4.70 is also used in section 5.3 in the proofs of propositions 3.17 and 5.20. However, these proofs do not depend on this conjecture, but are merely simplified by it.

4.4 The gluing theorem and symmetry

Definition 4.49 (Reversed mirror)**.** Let T be on oriented Conway tangle. Let $\mathrm{r}(T)$ be the tangle obtained by reversing the orientation of all components of T.
Furthermore, we write $\mathrm{mr}(T)$ for $\mathrm{m}(\mathrm{r}(T)) = \mathrm{r}(\mathrm{m}(T))$ and call it the **reversed mirror of** T.

Theorem 4.50 ([Zib20, Theorem 5.9])**.** *Let $L = T_1 \cup T_2$ be the union of two oriented Conway tangles T_1 and T_2 such that their orientations match. Then*

$$\widehat{\mathrm{HFK}}(L) \otimes V \;\;\cong\;\; \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(T_1)),\, \mathrm{HFT}(T_2))$$

if L is a knot and

$$\widehat{\mathrm{HFK}}(L) \;\;\cong\;\; \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(T_1)),\, \mathrm{HFT}(T_2))$$

otherwise.

Lemma 4.51 (Minimal intersection lemma)**.** *Let ${}^x/_y$, ${}^p/_q \in \mathbb{Q}\mathrm{P}^1$ be different. Then $\mathfrak{r}({}^x/_y)$ and $\mathfrak{r}({}^p/_q)$ intersect minimally if*

$$|\mathfrak{r}({}^x/_y) \cap \mathfrak{r}({}^p/_q)| = 2|xq - yp|.$$

Proof. From theorem 5.4 we know that $\mathfrak{r}(s) = \mathrm{HFT}(Q_s)$ for any $s \in \mathbb{Q}\mathrm{P}^1$. Let $L \coloneqq Q_{{}^p/_q}({}^x/_y) = Q_{-{}^x/_y} \cup Q_{{}^p/_q}$. We use lemma 2.51 and get that

$$L = Q_0 \cup Q_{\tau\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]}$$

for

$$\tau \coloneqq \begin{bmatrix} -b & -a \\ -x & y \end{bmatrix} = \begin{bmatrix} b & a \\ x & -y \end{bmatrix} \in \mathrm{Mod}(S^2_4)$$

where $a, b \in \mathbb{Z}$. We now apply the gluing theorem (minding corollary 2.29) to get

$$\mathrm{HF}(\mathrm{HFT}(Q_{{}^x/_y}), \mathrm{HFT}(Q_{{}^p/_q})) \;\;\cong\;\; \widehat{\mathrm{HFK}}(L) \;\;\cong\;\; \mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{\tau\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]})).$$

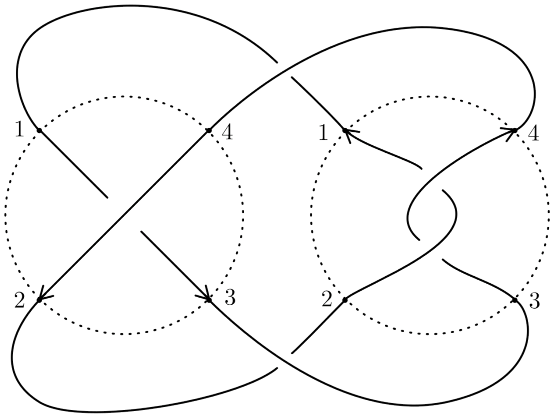


Figure 27: The left-handed trefoil $Q_1 \cup Q_{1/2}$

As ${}^x/_y \neq {}^p/_q$ we get that $\mathrm{HFT}(Q_{x/y})$ and $\mathrm{HFT}(Q_{p/q})$ are not homotopic. As τ is an automorphism we know that $\tau[\begin{smallmatrix} q \\ p \end{smallmatrix}] \neq [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ and hence $\mathrm{HFT}(Q_0)$ and $\mathrm{HFT}(\tau[\begin{smallmatrix} q \\ p \end{smallmatrix}])$ are also not homotopic. Hence

$$\dim \mathrm{HF}(\mathrm{HFT}(Q_{x/y}), \mathrm{HFT}(Q_{p/q}))$$

is the minimal intersection number between $\mathfrak{r}({}^x/_y)$ and $\mathfrak{r}({}^p/_q)$ and

$$\dim \mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{\tau[\begin{smallmatrix} q \\ p \end{smallmatrix}]})$$

is the minimal intersection number between $\mathfrak{r}(0)$ and $\mathfrak{r}(\tau[\begin{smallmatrix} q \\ p \end{smallmatrix}])$. From lemma 4.34 we get that

$$\dim \mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{\tau[\begin{smallmatrix} q \\ p \end{smallmatrix}]})$$

is given by

$$2|xq - yp|.$$

□

Example 4.52 (The left handed trefoil knot)**.** Take a look at the left handed trefoil knot $K = Q_1 \cup Q_{1/2}$ in figure 27. To easily see that this is actually the trefoil we can use remark 2.20 and proposition 2.30 to get

$$Q_1 \cup Q_{1/2} = \tau_2^{-1} Q_1 \cup \tau_2 Q_{1/2} = Q_\infty \cup Q_{1/3}.$$

The gluing theorem 4.50 gives us

$$\widehat{\mathrm{HFK}}(K) \otimes V \;\; \cong \;\; \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(Q_1),\, \mathrm{HFT}(Q_{1/2}))).$$

Minding corollary 2.29 we get that the curves $\mathrm{HFT}(\mathrm{mr}(Q_1)) = \mathrm{HFT}(Q_{-1})$ and $\mathrm{HFT}(Q_{1/2})$ are the rational curves depicted in figure 23. In example 4.44 we already computed the Lagrangian Floer homology of this pairing (figure 26), hence we get

$$\widehat{\mathrm{HFK}}(K) \otimes V \cong \delta^1(t^{-3/2}\mathbb{F}_2 \oplus t^{-1/2}\mathbb{F}_2^2 \oplus t^{1/2}\mathbb{F}_2^2 \oplus t^{3/2}\mathbb{F}_2).$$

The knot Floer homology is therefore given by

$$\widehat{\mathrm{HFK}}(K) \cong \delta^1(t^{-1}\mathbb{F}_2 \oplus t^0\mathbb{F}_2 \oplus t^1\mathbb{F}_2).$$

Compare this with the results in example 4.18. Interestingly, the relative isomorphism in this case is *absolute*. In the following, we will offer a justification why it is absolute with respect to Alexander grading. *The presumption is that there would also have to be a reason for the delta grading.*

Remark 4.53. (The problem of relativity) The isomorphism from the gluing theorem 4.50 is a priori relative, i.e. an isomorphism of bigraded vector spaces up to grading shifts. Also this is sufficient to prove many results, it is however not in our case. Because we will decompose the Lagrangian Floer homology into a direct sum later, we need this isomorphism to be absolute with respect to the Alexander grading. The core idea to solve this issue is by "symmetrizing" the absolute Alexander grading on the multicurves. This idea, which will consume the remaining section, has originated from remarks 2.2 and 2.3 in [Var21]. We will only be concerned about rational curves.

Definition 4.54 (Symmetric grading of vector spaces)**.** We call a graded vector space V **symmetrically graded** if

$$\dim V_i = \dim V_{-i}$$

for all $i \in \mathbb{Z}$.

Remark 4.55 ($\widehat{\mathrm{HFK}}$ is Alexander symmetric)**.** Ozsváth and Szabó showed in [OS04c] that $\widehat{\mathrm{HFK}}(L)$ of an oriented link L is symmetrically graded with respect to the Alexander grading.

Lemma 4.56. *Let V and W be graded vector spaces, V finite-dimensional and $f : V \to W$ be a relative isomorphism. If V and W are symmetrically graded, f must be an (absolute) isomorphism.*

Proof. Let $M := \max\{i \in \mathbb{Z} \mid V_i \neq 0\}$. Because f is a relative isomorphism, there exists $s \in \mathbb{Z}$ such that $f(V_i) = W_{i+s}$ for all $i \in \mathbb{Z}$. Assume $s > 0$ then we know, that $W_{M+s} \neq 0$ and because of the symmetry that $W_{-(M+s)} \neq 0$. Therefore $V_{-(M+s)-s} \neq 0$, but because of symmetry $V_{-M-2s} = V_{M+2s} = 0$ by definition of M. The analogue argument works for $s < 0$. □

Lemma 4.57. *Let $L = T_1 \cup T_2$ be the union of two oriented Conway tangles T_1 and T_2 such that their orientations match. If*

$$\mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(T_1)), \mathrm{HFT}(T_2))$$

is symmetrically Alexander graded (for fixed absolute bigradings of the two multicurves) then the isomorphism from Theorem 4.50 is absolute with respect to the Alexander grading.

Proof. From remark 4.55 we know that $\widehat{\mathrm{HFK}}(L)$ and hence also its stabilization $\widehat{\mathrm{HFK}}(L) \otimes V$ have symmetric Alexander grading (and are finite-dimensional). As the gluing theorem gives us an relative isomorphism with respect to the Alexander grading lemma 4.56 shows the claim. □

Definition 4.58. Let T be an oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$ absolutely bigraded. We define for $s \in \{a, b, c, d\}$

$$\textstyle\sum_s(\gamma) := \sum_{x \in X_s(\gamma)} A(x) \quad \in \frac{1}{2}\mathbb{Z}$$

to be the **sum over all Alexander gradings on the site** s and $V_s(\gamma)$ to be the Alexander graded vector space

$$V_s(\gamma) := \mathbb{F}_2\langle X_s(\gamma)\rangle$$

where the generators of $V_s(\gamma)$ have the same Alexander grading as the generators on the site s.

Lemma 4.59. *Let $Q_{p/q}$, $p/q \in \mathbb{Q}\mathrm{P}^1$ be an oriented rational tangle and $\gamma := \mathrm{HFT}(Q_{p/q})$. Then*

$$N_{|}(\gamma) := |X_a(\gamma)| = |X_c(\gamma)| = |q|$$

and

$$N_{-}(\gamma) := |X_b(\gamma)| = |X_d(\gamma)| = |p|.$$

Proof. This follows from the considerations of remark 4.33. □

Definition 4.60 (Symmetric Alexander grading)**.** Let T be an oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$ absolutely Alexander graded. We call γ **symetrically Alexander graded** if

$$\textstyle\sum_a(\gamma) + \sum_c(\gamma) = \sum_b(\gamma) + \sum_d(\gamma) = 0.$$

Such a symmetric Alexander grading is (if existent) unique, as the relative Alexander grading of $\mathrm{HFT}(\gamma)$ is unique up to grading shifts.

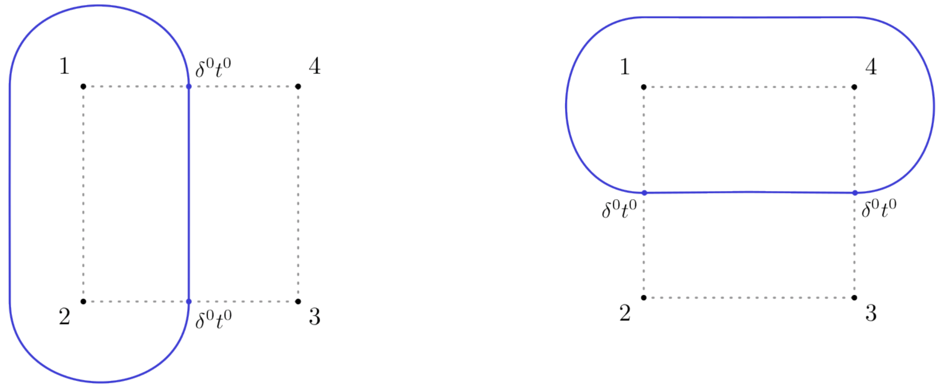
Example 4.61. In figure 28 we see the unique symmetric Alexander gradings of $\mathrm{HFT}(Q_\infty)$ and $\mathrm{HFT}(Q_0)$ (independent of the chosen orientation). In example 23 we have also already seen the symmetric Alexander grading of $\mathrm{HFT}(Q_{-1})$ and $\mathrm{HFT}(Q_{1/2})$ for the given orientations.

Lemma 4.62. *Let Q be an oriented rational tangle and $\gamma := \mathrm{HFT}(Q)$ absolutely Alexander graded. Then, if $Q \neq Q_0$, $Q(0)$ exists and $\gamma_0 := \mathrm{HFT}(Q_0)$ is symmetrically Alexander graded,*

$$V_b(\gamma) \otimes V_* = \mathrm{HF}_*(\gamma_0, \gamma) = V_d(\gamma) \otimes V_*.$$

In the same manner, if $Q \neq Q_\infty$, $Q(\infty)$ exists and $\gamma_\infty := \mathrm{HFT}(Q_\infty)$ is symmetrically Alexander graded, then

$$V_a(\gamma) \otimes V_* = \mathrm{HF}_*(\gamma_\infty, \gamma) = V_c(\gamma) \otimes V_*.$$



(a) $\mathrm{HFT}(Q_\infty)$ (b) $\mathrm{HFT}(Q_0)$

Figure 28: Trivial rational curves

Proof. We do the argument for γ_0, but γ_∞ works similarly. Let $s \in \{b, d\}$ and let $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ be the slope of $\mathbb{Q}$. Homotope γ_0 to lie close around s as in figure 29 and such that

$$|\gamma_0 \cap \gamma| = 2|p|. \tag{$*$}$$

This is possible by lemma 4.59. In this case lemma 4.51 implies that that γ_0 and γ intersect minimally. As γ_0 and γ are not homotopic (by the assumption $Q \neq Q_0$) we can compute the Lagrangian Floer homology from the minimal intersection points. As γ intersects the sites minimally by assumption all intersections $\gamma_0 \cap \gamma$ look (up to homotopy) like in figure 30 (γ is red, γ_0 is blue). For every generator $x \in X_s(\gamma)$ we compute the Alexander grading of the generators $x_1, x_2 \in \gamma_0 \cap \gamma$ using the basic connecting domains coloured green and red:

$$A(x_1) = A(x) - A(y) \pm 1/2 = A(x_1) \pm 1/2$$
$$A(x_2) = A(x) - A(y) \mp 1/2 = A(x_2) \mp 1/2$$

Mind that $A(y) = 0$ as γ_0 is symmetrically Alexander graded (Figure 28). The signs of $\pm 12, \mp 12$ depend on the direction the puncture is pointing to, but the signs are always different. The construction and ($*$) imply that all generators of $\mathrm{HF}(\gamma_0, \gamma)$ are given in this way, hence

$$\mathrm{HF}_*(\gamma_0, \gamma) = V_s(\gamma) \otimes V_*.$$

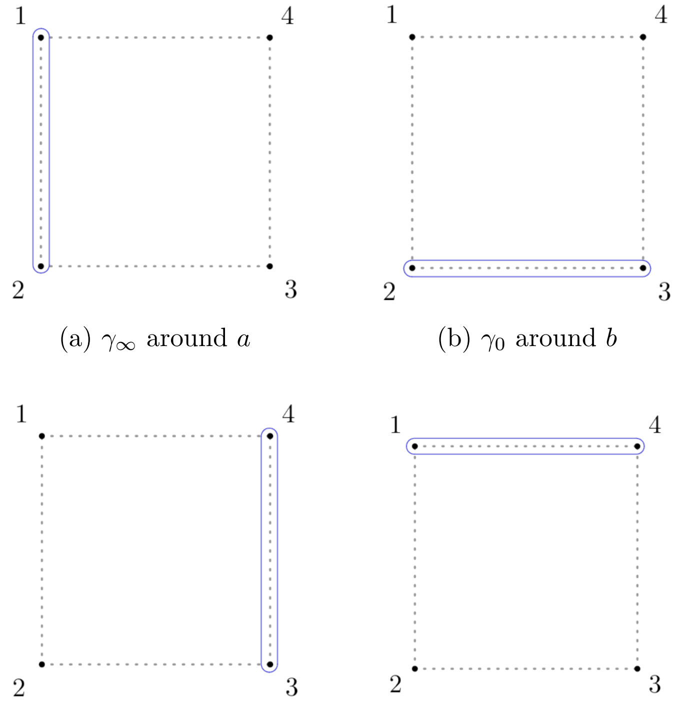
□

Corollary 4.63. *Let Q be an oriented rational tangle and $\gamma := \mathrm{HFT}(Q)$ absolutely Alexander graded. If $Q(0)$ exists, then*

$$V_-(\gamma) := V_b(\gamma) = V_d(\gamma) \quad \textit{and} \quad \textstyle\sum_-(\gamma) := \sum_b(\gamma) = \sum_d(\gamma).$$

If $Q(\infty)$ exists, then

$$V_|(\gamma) := V_a(\gamma) = V_c(\gamma) \quad \textit{and} \quad \textstyle\sum_|(\gamma) := \sum_a(\gamma) = \sum_c(\gamma).$$



(a) γ_∞ around a (b) γ_0 around b (c) γ_∞ around c (d) γ_0 around d

Figure 29: Idea of proof 4.62

Proof. For the trivial cases $Q \in \{Q_0, Q_\infty\}$ this can be checked (Figure 28) and for all other cases we get the equalities from lemma 4.62. □

Corollary 4.64. *Let Q be an oriented rational tangle and $\gamma := \mathrm{HFT}(Q)$ be symmetrically Alexander graded. If $Q(0)$ exists, then*

$$\textstyle\sum_{-}(\gamma) = 0$$

and V_- is symmetrically graded with respect to the Alexander grading. If $Q(\infty)$ exists, then

$$\textstyle\sum_{|}(\gamma) = 0$$

and $V_|$ is symmetrically graded with respect to the Alexander grading.

Proof. Let $Q(0)$ be existent. By definition 4.60

$$\textstyle\sum_b(\gamma) + \sum_d(\gamma) = 0.$$

as γ is symmetrically Alexander graded. By corollary 4.63

$$\textstyle\sum_b(\gamma) + \sum_d(\gamma) = 2\sum_{-}(\gamma)$$

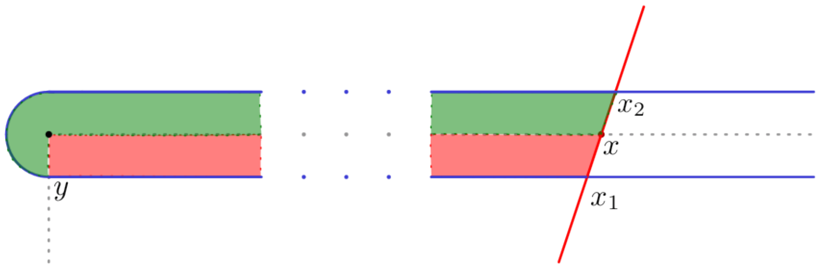


Figure 30: Computation of $\mathrm{HF}(\gamma_0, \gamma)$

hence $\sum_-(\gamma) = 0$. Furthermore, lemma 4.62 gives

$$V_-(\gamma) \otimes V_* = \mathrm{HF}_*(\gamma_0, \gamma)$$

and theorem 4.50 implies

$$\mathrm{HF}_*(\gamma_0, \gamma) \cong \widehat{\mathrm{HFK}}_*(Q(0)).$$

As $\widehat{\mathrm{HFK}}_*(Q(0))$ is symmetrically Alexander graded (remark 4.55) the vector space V_- must by symmetrically graded up to a shift. However, $\sum_-(\gamma) = 0$ implies that this shift must be zero.

The analogue argument works for $Q(\infty)$ existent. $\square$

Definition 4.65. For $s \in \mathbb{Q}\mathrm{P}^1$ we define

$$\mathrm{sgn}(s) := \begin{cases} 0, & \text{if } s \in \{0, \infty\}, \\ 1, & \text{if } s \in \mathbb{Q}, s > 0, \\ -1, & \text{if } s \in \mathbb{Q}, s < 0. \end{cases}$$

Lemma 4.66. *Let $Q_{p/q}$ be an oriented rational tangle and $\mathrm{HFT}(Q_{p/q})$ be symmetrically Alexander graded. If $Q(\infty)$ exists, then*

$$\mathrm{sgn}(p/q)\,\mathrm{sgn}((p+q)/q) \in \{0,1\} \implies \tau_1 \,\mathrm{HFT}(Q) \text{ is symetrically Alexander graded,}$$
$$\mathrm{sgn}(p/q)\,\mathrm{sgn}((p-q)/q) \in \{0,1\} \implies \tau_1^{-1} \,\mathrm{HFT}(Q) \text{ is symetrically Alexander graded.}$$

If $Q(0)$ exists, then

$$\mathrm{sgn}(p/q)\,\mathrm{sgn}(p/(q+p)) \in \{0,1\} \implies \tau_2 \,\mathrm{HFT}(Q) \text{ is symetrically Alexander graded,}$$
$$\mathrm{sgn}(p/q)\,\mathrm{sgn}(p/(q-p)) \in \{0,1\} \implies \tau_2^{-1} \,\mathrm{HFT}(Q) \text{ is symetrically Alexander graded.}$$

Proof. Observe that for $\tau \in \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}$

$$\tau \,\mathrm{HFT}(Q_{p/q}) \overset{4.27}{=} \mathrm{HFT}(\tau Q_{p/q}) \overset{2.30}{=} \mathrm{HFT}(Q_{\tau \cdot \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]}).$$

Hence lemma 4.59 shows for $\gamma := \mathrm{HFT}(Q_{p/q})$ that:

$$\begin{aligned} N_|(\tau_1\gamma) &= |q| \quad \text{and} \quad N_-(\tau_1\gamma) = |p+q| \\ N_|(\tau_1^{-1}\gamma) &= |q| \quad \text{and} \quad N_-(\tau_1^{-1}\gamma) = |p-q| \\ N_|(\tau_2\gamma) &= |q+p| \quad \text{and} \quad N_-(\tau_2\gamma) = |p| \\ N_|(\tau_2^{-1}\gamma) &= |q-p| \quad \text{and} \quad N_-(\tau_2^{-1}\gamma) = |p| \end{aligned}$$

We want to show that

$$\textstyle\sum_a(\tau\gamma) + \sum_c(\tau\gamma) = \sum_b(\tau\gamma) + \sum_d(\tau\gamma) = 0$$

is satisfied. For that we use remark 4.42: We make the argument for the case that $Q(\infty)$ exists and $\mathrm{sgn}(p/q)\,\mathrm{sgn}((p+q)/q) \in \{0,1\}$, but the other cases work similarly.

First, mind that the sum $\sum_a(\tau\gamma) + \sum_c(\tau\gamma)$ is unchanged by τ_1. We therefore only need to look at the sites b and d. We know that the site c has $|q|$ generators, hence both b and d get additional $|q|$ provisional generators. However, because of the above equalities concerning the number of generators and the sign assumption, either *all* provisional generators cancel with existing generators or *all* provisional generators stay (this depends on the signs of p/q).

If all cancel, they must cancel with existing generators of the same Alexander grading. In formulas this means

$$\textstyle\sum_b(\tau_1\gamma) = \sum_b(\gamma) - (\sum_c(\gamma) + |q|\lambda/2)$$

and

$$\textstyle\sum_d(\tau_1\gamma) = \sum_d(\gamma) - (\sum_c(\gamma) - |q|\lambda/2)$$

where the λ is defined as in remark 4.42. In sum we have

$$\begin{aligned} \textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) &= \textstyle\sum_b(\gamma) - (\sum_c(\gamma) + |q|\lambda/2) + \sum_d(\gamma) - (\sum_c(\gamma) - |q|\lambda/2) \\ &\overset{(*)}{=} \textstyle\sum_b(\gamma) + \sum_d(\gamma) - 2\sum_c(\gamma) \\ &= \textstyle\sum_b(\gamma) + \sum_d(\gamma) \\ &= 0 \end{aligned}$$

where $(*)$ holds, because corollary 4.64 shows $\sum_c(\gamma) = 0$.

If all provisional generators stay, we get in formulas

$$\textstyle\sum_b(\tau_1\gamma) = \sum_b(\gamma) + (\sum_c(\gamma) + |q|\lambda/2)$$

and

$$\textstyle\sum_d(\tau_1\gamma) = \sum_d(\gamma) + (\sum_c(\gamma) - |q|\lambda/2).$$

In sum we have

$$\begin{aligned} \textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) &= \textstyle\sum_b(\gamma) + (\sum_c(\gamma) + |q|\lambda/2) + \sum_d(\gamma) + (\sum_c(\gamma) - |q|\lambda/2) \\ &\overset{(**)}{=} \textstyle\sum_b(\gamma) + \sum_d(\gamma) + 2\sum_c(\gamma) \\ &= \textstyle\sum_b(\gamma) + \sum_d(\gamma) \\ &= 0 \end{aligned}$$

where $(**)$ holds as above, because corollary 4.64 shows $\sum_c(\gamma) = 0$. □

Lemma 4.67. *Let $Q_{p/q}$ be an oriented rational tangle and $\mathrm{HFT}(Q_{p/q})$ be symmetrically Alexander graded. If $Q(\infty)$* does not *exists, then*

$$\operatorname{sgn}(p/q)\operatorname{sgn}((p+q)/q) \in \{0,1\} \implies \tau_1 \,\mathrm{HFT}(Q) \textit{ is symetrically Alexander graded,}$$
$$\operatorname{sgn}(p/q)\operatorname{sgn}((p-q)/q) \in \{0,1\} \implies \tau_1^{-1} \,\mathrm{HFT}(Q) \textit{ is symetrically Alexander graded.}$$

If $Q(0)$ does not *exists, then*

$$\operatorname{sgn}(p/q)\operatorname{sgn}(p/(q+p)) \in \{0,1\} \implies \tau_2 \,\mathrm{HFT}(Q) \textit{ is symetrically Alexander graded,}$$
$$\operatorname{sgn}(p/q)\operatorname{sgn}(p/(q-p)) \in \{0,1\} \implies \tau_2^{-1} \,\mathrm{HFT}(Q) \textit{ is symetrically Alexander graded.}$$

Proof. Observe that for $\tau \in \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}$

$$\tau\,\mathrm{HFT}(Q_{p/q}) \overset{4.27}{=} \mathrm{HFT}(\tau Q_{p/q}) \overset{2.30}{=} \mathrm{HFT}(Q_{\tau\cdot\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]}).$$

Hence lemma 4.59 shows for $\gamma := \mathrm{HFT}(Q_{p/q})$ that:

$$\begin{aligned} N_|(\tau_1\gamma) &= |q| \quad \text{and} \quad N_-(\tau_1\gamma) = |p+q| \\ N_|(\tau_1^{-1}\gamma) &= |q| \quad \text{and} \quad N_-(\tau_1^{-1}\gamma) = |p-q| \\ N_|(\tau_2\gamma) &= |q+p| \quad \text{and} \quad N_-(\tau_2\gamma) = |p| \\ N_|(\tau_2^{-1}\gamma) &= |q-p| \quad \text{and} \quad N_-(\tau_2^{-1}\gamma) = |p| \end{aligned}$$

We want to show that

$$\textstyle\sum_a(\tau\gamma) + \sum_c(\tau\gamma) = \sum_b(\tau\gamma) + \sum_d(\tau\gamma) = 0$$

is satisfied. For that we use remark 4.42: We make the argument for the case that $Q(\infty)$ does not exists and $\operatorname{sgn}(p/q)\operatorname{sgn}((p+q)/q) \in \{0,1\}$, but the other cases work similarly.

First, mind that the sum $\sum_a(\tau\gamma)+\sum_c(\tau\gamma)$ is unchanged by τ_1. We therefore only need to look at the sites b and d. We know that the site c has $|q|$ generators, hence both b and d get additional $|q|$ provisional generators. However, because of the above equalities concerning the number of generators and the sign assumption, either *all* provisional generators cancel with existing generators or *all* provisional generators stay (this depends on the signs of p/q).

If all cancel, they must cancel with existing generators of the same Alexander grading. In formulas this means

$$\textstyle\sum_b(\tau_1\gamma) = (\sum_b(\gamma) + |p|\lambda/2) - \sum_c(\gamma)$$

and

$$\textstyle\sum_d(\tau_1\gamma) = (\sum_d(\gamma) + |p|\lambda/2) - \sum_c(\gamma)$$

where the λ is defined as in remark 4.42. In sum we have

$$\begin{aligned}\textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) &= (\textstyle\sum_b(\gamma) + |p|\lambda/2) - \sum_c(\gamma) + (\sum_d(\gamma) + |p|\lambda/2) - \sum_c(\gamma)\\ &= \textstyle\sum_b(\gamma) + \sum_d(\gamma) + |p|\lambda - 2\sum_c(\gamma)\\ &= |p|\lambda - 2\textstyle\sum_c(\gamma).\end{aligned}$$

We now have to determine $\sum_c(\gamma)$. Because $Q(\infty)$ does not exists, the closure $Q(0)$ must exist. Remember that we are in the case that all provisional generators cancel for τ_1, which means that $p/q < 0$ must hold. This implies that the case

$$\operatorname{sgn}(p/q)\operatorname{sgn}(p/(q-p)) \in \{0,1\} \implies \tau_2^{-1}\gamma \text{ is symetrically Alexander graded}$$

from lemma 4.66 applies. Mind that $\tau_2^{-1}Q$ has an existing ∞-closure, hence

$$\textstyle\sum_c(\tau_2^{-1}\gamma) = 0$$

by corollary 4.64. Furthermore from remark 4.42 we get that

$$\textstyle\sum_c(\tau_2^{-1}\gamma) = \sum_c(\gamma) + (\sum_b(\gamma) - |p|\lambda/2)$$

and $\sum_b(\gamma) = 0$ by corollary 4.64, hence

$$\textstyle\sum_c(\gamma) = |p|\lambda/2.$$

In total we have

$$\textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) = |p|\lambda - 2\sum_c(\gamma) = |p|\lambda - 2(|p|\lambda/2) = 0.$$

If all provisional generators stay, we get in formulas

$$\textstyle\sum_b(\tau_1\gamma) = (\sum_b(\gamma) + |p|\lambda/2) + \sum_c(\gamma)$$

and

$$\textstyle\sum_d(\tau_1\gamma) = (\sum_d(\gamma) + |p|\lambda/2) + \sum_c(\gamma).$$

In sum we have

$$\begin{aligned}\textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) &= (\textstyle\sum_b(\gamma)|p|\lambda/2) + \sum_c(\gamma) + (\sum_d(\gamma)|p|\lambda/2) + \sum_c(\gamma)\\ &= \textstyle\sum_b(\gamma) + \sum_d(\gamma) + |p|\lambda + 2\sum_c(\gamma)\\ &= |p|\lambda + 2\textstyle\sum_c(\gamma).\end{aligned}$$

We now have to determine $\sum_c(\gamma)$. Because $Q(\infty)$ does not exists, the closure $Q(0)$ must exist. Remember that we are in the case that all provisional generators stay, which means that $p/q \geq 0$ must hold. This implies that the case

$$\operatorname{sgn}(p/q)\operatorname{sgn}(p/(q+p)) \in \{0,1\} \implies \tau_2\gamma \text{ is symetrically Alexander graded}$$

from lemma 4.66 has to apply. Mind that $\tau_2 Q$ has an existing ∞-closure, hence

$$\textstyle\sum_c(\tau_2\gamma) = 0$$

by corollary 4.64. Furthermore from remark 4.42 we get that

$$\textstyle\sum_c(\tau_2\gamma) = \sum_c(\gamma) + (\sum_b(\gamma) + |p|\lambda/2)$$

and $\sum_b(\gamma) = 0$ by corollary 4.64, hence

$$\textstyle\sum_c(\gamma) = -|p|\lambda/2.$$

In total we have

$$\textstyle\sum_b(\tau\gamma) + \sum_d(\tau\gamma) = |p|\lambda - 2\sum_c(\gamma) = |p|\lambda + 2(-|p|\lambda/2) = 0.$$

□

Theorem 4.68. *Let Q be a oriented rational tangle. Then* $\mathrm{HFT}(Q)$ *has a symmetric Alexander grading.*

Proof. Let s be the fraction of Q. We find a continued fraction expansion of s where all coefficients have the same sign (same expansion as in proof 2.49) hence Conway's algorithm 2.25 shows that

$$Q_s := \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd,} \end{cases} \tag{$*$}$$

where all a_i's have the same sign (with a_1 possible being zero). From theorem 4.27 we get for $\gamma := \mathrm{HFT}(Q_{p/q})$, $\gamma_\infty := \mathrm{HFT}(Q_\infty)$, $\gamma_0 := \mathrm{HFT}(Q_0)$ that

$$\gamma := \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\gamma_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\gamma_0, & \text{if } n \text{ is odd.} \end{cases}$$

The core idea is that there appear no cancellations between provisional and existing generators after each single $\tau_1^{\pm 1}$, $\tau_2^{\pm 1}$.

Let $s_i \in \mathbb{Q}\mathrm{P}^1$ for $i \in \mathbb{N}$ be the fraction of the rational tangle resulting after the first (from right to left) i actions in $(*)$. As all coefficients have the same sign (or are zero) we get that

$$\operatorname{sgn}(s_i)\operatorname{sgn}(s_{i+1}) \in \{0, 1\}$$

for all (possible) $i \in \mathbb{N}$. This can be easily checked using proposition 2.30. Hence lemmas 4.66 and 4.67 show that γ is symmetrically Alexander graded if γ_∞ respectively γ_0 is symmetrically Alexander graded. Both have such a grading illustrated in figure 28. □

Proposition 4.69. *Let Q be a oriented rational tangle and $\tau \in \mathrm{Mod}(S^2_4(Q))$. If* $\mathrm{HFT}(Q)$ *is symmetrically Alexander graded then τ* $\mathrm{HFT}(Q)$ *is symmetrically Alexander graded.*

Proof. As τ_1, τ_2 generate $\mathrm{Mod}(S^2_4(Q))$ we find $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{Z} \setminus \{0\}$ for $i \in \{1, \ldots, n\}$ such that

$$\tau = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore it is sufficient to proof the claim for $\tau \in \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}$. Let $\gamma := \mathrm{HFT}(Q)$. By definition we know

$$\textstyle\sum_a(\gamma) + \sum_c(\gamma) = \sum_b(\gamma) + \sum_d(\gamma) = 0.$$

From remark 4.42 we get that one of the two sums is unchanged by τ, i.e.

$$\textstyle\sum_a(\tau\gamma) + \sum_c(\tau\gamma) = 0 \quad \vee \quad \sum_b(\tau\gamma) + \sum_d(\tau\gamma) = 0. \tag{$*$}$$

From proposition 2.30 we get that τQ is still a rational tangle hence

$$\mathrm{HFT}(\tau Q)$$

can be symmetrically Alexander graded by theorem 4.68. The relative equality

$$\tau\,\mathrm{HFT}(Q) = \mathrm{HFT}(\tau Q)$$

from theorem 4.27 shows that $\tau\gamma$ can be symmetrically Alexander graded too. As one of the sums ($*$) is zero we get by the uniqueness of the symmetric Alexander grading that $\tau\gamma$ must already be symmetrically Alexander graded (any Alexander grading shift would change the zero sum to something not zero). □

Proposition 4.70. *Let $L = Q_s \cup Q_t$ be an oriented rational link. If we fix the symmetric Alexander grading on* $\mathrm{HFT}(\mathrm{mr}(Q_s))$ *and* $\mathrm{HFT}(Q_t)$ *then*

$$\mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(Q_s)),\, \mathrm{HFT}(Q_t))$$

is symmetrically Alexander graded.

Proof. Using $\pi \in \mathrm{Mod}(S^2_4)$ from remark 3.19 we have

$$\begin{aligned} \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(Q_s)),\, \mathrm{HFT}(Q_t)) &\overset{4.47}{=} \mathrm{HF}(\pi\,\mathrm{HFT}(\mathrm{mr}(Q_s)),\, \pi\,\mathrm{HFT}(Q_t)) \\ &\overset{4.27}{=} \mathrm{HF}(\mathrm{HFT}(\pi\,\mathrm{mr}(Q_s)),\, \mathrm{HFT}(\pi Q_t)) \\ &\overset{2.20}{=} \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}(\pi^{\mathrm{m}} Q_s)),\, \mathrm{HFT}(\pi Q_t)) = \ldots \end{aligned}$$

Case I: n is even. Remark 3.19 shows that $\pi Q_t(\infty)$ must exist.

$$\ldots = \mathrm{HF}(\mathrm{HFT}(Q_\infty),\, \mathrm{HFT}(\pi Q_t)) = \ldots$$

By lemma 4.69 both curves $\mathrm{HFT}(Q_\infty)$ and $\mathrm{HFT}(\pi Q_t)$ are symmetrically Alexander graded, hence lemma 4.62 implies

$$\begin{aligned} \ldots &= \mathrm{HF}(\mathrm{HFT}(Q_\infty),\, \mathrm{HFT}(\pi Q_t)) = \ldots \\ &= V_|(\mathrm{HFT}(\pi Q_t)) \otimes V_*. \end{aligned}$$

and by corollary 4.64 this vector space is symmetrically graded.
Case II: n is odd. Remark 3.19 shows that $\pi Q_t(0)$ must exist.

$$\ldots = \mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(\pi Q_t)) = \ldots$$

By lemma 4.69 both curves $\mathrm{HFT}(Q_0)$ and $\pi\,\mathrm{HFT}(Q_t)$ have are symmetrically Alexander graded, hence lemma 4.62 implies

$$\begin{aligned}\ldots &= \mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(\pi Q_t)) = \ldots \\ &= V_-(\mathrm{HFT}(\pi Q_t)) \otimes V_*\end{aligned}$$

and by corollary 4.64 this vector space is symmetrically graded. □

5 Knot Floer homology of two-bridge links

As all rational links (except the unlink) are δ-thin (4.17), we can compute their knot Floer homology (up to a δ-grading shift) using their Alexander polynomial (4.13). There are elegant formulas for the Alexander polynomial of rational links by Hartley, Minkus and Hoste which can be interpreted as walks on integer lattices ([Hos19]). Yet, out of curiosity we develop our own formulas for computing the knot Floer homology and then deduce the formula for the univariant Alexander polynomial with our methods.

Furthermore, we prove some results concerning cyclical even continued fractions of rational links (in particular proposition 3.17). On another note, the author has made software implementations of this chapter. In particular an efficient way to compute the knot Floer homology (up to a δ-grading shift) for any oriented rational link (see appendix A).

5.1 General formula

Definition 5.1. For $s \in \mathbb{Q}\mathrm{P}^1$ and $k \in \mathbb{N}$ define

$$\Delta_k(s) \coloneqq -\mathrm{sgn}(s) \cdot \left(k \cdot (1,1,0,0) + 2 \cdot \sum_{y=1}^{k} (\Gamma_y(s) + \Gamma_{y-1}(s)) \right)$$

where

$$\Gamma_y(s) \coloneqq \begin{cases} (\lfloor \frac{\lambda_y(s)}{2} \rfloor, 0, 0, \lceil \frac{\lambda_y(s)}{2} \rceil), & \text{if } y \text{ is odd,} \\ (0, \lfloor \frac{\lambda_y(s)}{2} \rfloor, \lceil \frac{\lambda_y(s)}{2} \rceil, 0), & \text{if } y \text{ is even,} \end{cases}$$

and

$$\lambda_y(s) \coloneqq \lfloor y \cdot |s^{-1}| \rfloor \quad \text{for } y \in \mathbb{N}.$$

Proposition 5.2. *Let $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ and $Q_{p/q}$ oriented with ordered matching P. Let $\mathrm{HFT}(Q_{p/q})$ be absolutely Alexander graded. Define the Alexander graded vector spaces*

$$H_P^b(p/q) \coloneqq \mathbb{F}_2\langle \{ \mathrm{u}_P(\Delta_{2k}(p/q)) \mid k \in \{0, \ldots, p-1\} \} \rangle$$

and

$$H^d_P({}^p/_q) \coloneqq \mathbb{F}_2\langle\{\mathrm{u}_P(\Delta_{2k+1}({}^p/_q)) \mid k \in \{0, \ldots, p-1\}\}\rangle$$

then

$$V_b(\mathrm{HFT}(Q_{{}^p/_q})) \cong H^b_P({}^p/_q) \quad \textit{and} \quad V_d(\mathrm{HFT}(Q_{{}^p/_q})) \cong H^d_P({}^p/_q).$$

Proof. (This proof uses some not mentioned knowledge about the determination of the bigrading in the covering space, see [LMZ21, Section 3.6]) Let $\gamma \coloneqq \mathrm{HFT}(Q_{{}^p/_q})$. As ${}^p/_q \neq 0$ we know that $x \in \operatorname{im}\gamma \cap b$ exists. We can homotope γ such that:

1. γ lifts to a straight line $\tilde{\gamma}$ along η,
2. $\tilde{x} \coloneqq \tilde{\gamma}(0)$ is a lift of x along η,
3. $\tilde{x}$ "lies close" to a puncture labelled 2.

We need to specify the last point: Let $\tilde{b}$ be the lift of b such that $\tilde{x} \in \tilde{b}$. Let y be a point on $\tilde{b}$ between $\tilde{x}$ and the puncture adjacent to $\tilde{b}$ with the label 2. Then the map

$$(s,t) \mapsto \tilde{\gamma}(s) + t \cdot (y - \tilde{x})$$

is a homotopy between $\tilde{\gamma}$ and $\tilde{\gamma} + y - \tilde{x}$. Additionally a possible reparametrization of $\tilde{\gamma}$ allows us to assume that $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{R}^2$ has a non-negative first coordinate.

The idea is to compute the Alexander grading of all intersections

$$X \coloneqq \operatorname{im}\tilde{\gamma} \cap \eta^{-1}(b \cup d)$$

in the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Let $x_0 \coloneqq \tilde{x}$ and number the intersection points $x_i \in X$ for $i \in \{1, \ldots, 2p-1\}$ in order along $\tilde{\gamma}$. Let A be the vertical line given by the lift of the site a closest to $\tilde{x}$. We call B the *base line*. Futhermore, we define connecting domains φ_i from x_{i-1} to x_i for $i \in \{1, \ldots 2p-1\}$:

Let φ_i be the domain bordered by the base line B the curve $\tilde{\gamma}$ and the (punctured) horizontal lines on which x_{i-1} and x_i lie. We call this domains understandably *picket domains*. See figure 31 for an exemplary presentation. If ${}^p/_q > 0$ we give the regions of our picket domains a multiplicity of +1, and other a multiplicity of -1. Thereby φ_i is a connecting domain from x_{i-1} to x_i in both cases.

From [LMZ21, Lemma 3.17] we know that

$$\hat{A}(x_i) - \hat{A}(x_{i-1}) = -\hat{A}(\varphi_i)$$

hence we have to determine $\hat{A}(\varphi_i)$. Because of the particular form of our picket domains there are only three contributions (summands) to $\hat{A}(\varphi_i)$ all from punctures on the border of φ_i:

1. the punctures on the base line,
2. the punctures between the base line and x_i,

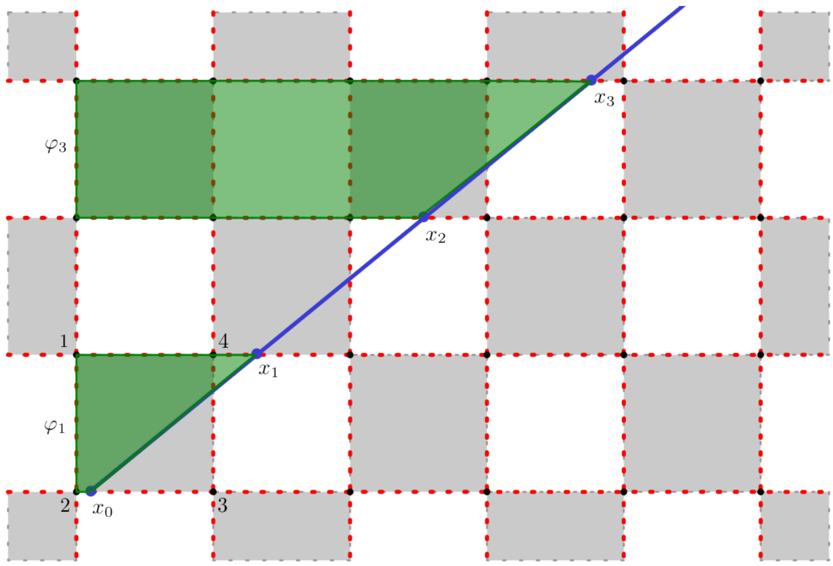


Figure 31: Picket domains

3. the punctures between the base line and x_{i-1}.

The base line punctures always have the same contribution

$$\operatorname{sgn}(p/q) \cdot (1,1,0,0)$$

independent of i. Next, we check that the number of punctures N_i between the base line and the point x_i is given by

$$N_i := \lfloor i \cdot |\frac{q}{p}| \rfloor.$$

This is ensured by the the three properties of γ above. The contribution of these punctures between the base line and x_i is dependent on the parity of i (puncture labels 1,4 or 2,3) but can be determined by

$$M_i := \operatorname{sgn}(p/q) \cdot 2 \cdot \begin{cases} (\lfloor \frac{N_i}{2} \rfloor, 0, 0, \lceil \frac{N_i}{2} \rceil), & \text{if } i \text{ is odd,} \\ (0, \lfloor \frac{N_i}{2} \rfloor, \lceil \frac{N_i}{2} \rceil, 0), & \text{if } i \text{ is even.} \end{cases}$$

A short explanation: Along the border from the base line to x_i the according label pair is alternating between the lables. Thus, if N_i is even, both appear $N_i/2$ times. If N_i is odd the puncture label 4 (resp. 3) appears exactly once more then 1 (resp. 2). The Alexander grading of our picket domain φ_i is then given by

$$\hat{A} = (1,1,0,0) + M_i + M_{i-1}.$$

We're almost done. We fix the Alexander grading $\hat{A}(x_0) = (0,0,0,0)$ and compute

$$\hat{A}(x_k) - \hat{A}(x_0) = -\sum_{i=1}^{k} A(\varphi_i)$$

which exactly gives us

$$\hat{A}(x_k) = \Delta_k({}^p\!/\!_q).$$

We have now computed the multivariate Alexander grading for all generators on the sites b and d. Because of $\mathrm{u}_P \circ \hat{A} = A$ the claim directly follows from the uniqueness of the relative bigrading on a single curve. □

Remark 5.3. In the situation of proposition 5.2 we know from corollary 4.63 that if $Q_{p/q}(0))$ exists,

$$V_b(\mathrm{HFT}(Q_{p/q})) = V_d(\mathrm{HFT}(Q_{p/q}))$$

hence we write

$$H_P({}^p\!/\!_q) \coloneqq H_P^b({}^p\!/\!_q) \cong H_P^d({}^p\!/\!_q)$$

and get

$$V_-(\mathrm{HFT}(Q_{p/q})) \cong H_P({}^p\!/\!_q).$$

Theorem 5.4 (Knot Floer homology of rational links). *Let $L = Q_0 \cup Q_{p/q}$ with ${}^p\!/\!_q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ be a oriented rational link and P be the ordered matching of $Q_{p/q}$. Let $H_P({}^p\!/\!_q)$ be as in the remark 5.3, then*

$$\widehat{\mathrm{HFK}}(L) \cong \delta^0 H_P({}^p\!/\!_q)$$

if L is a knot and

$$\widehat{\mathrm{HFK}}(L) \cong \delta^0 H_P({}^p\!/\!_q) \otimes V$$

otherwise.

Proof. From proposition 5.2 and lemma 4.62 we get

$$H_P({}^p\!/\!_q) \otimes V \cong \mathrm{HF}_*(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{p/q})).$$

The gluing theorem 4.50 gives

$$\mathrm{HF}_*(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{p/q})) \cong \widehat{\mathrm{HFK}}_*(L) \otimes V_*$$

if L is a knot and

$$\mathrm{HF}_*(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{p/q})) \cong \widehat{\mathrm{HFK}}_*(L)$$

otherwise. Furthermore, we know from remark 4.17 that $L = N({}^p\!/\!_q)$ is δ-thin. Therefore it is enough to support the Alexander graded vector spaces in a single δ-grading to get a relative isomorphism of bigraded vector spaces. □

Definition 5.5. For $s \in \mathbb{Q}\mathrm{P}^1$ and $k \in \mathbb{N}$ we define

$$G_k(s) \coloneqq \Delta_k(s) - \mathrm{sgn}(s) \cdot (\rho_k + 2\Gamma_k(s))$$

where

$$\rho_k \coloneqq \begin{cases} (1,0,0,0), & \text{if } k \text{ is odd,} \\ (0,1,0,0), & \text{if } k \text{ is even.} \end{cases}$$

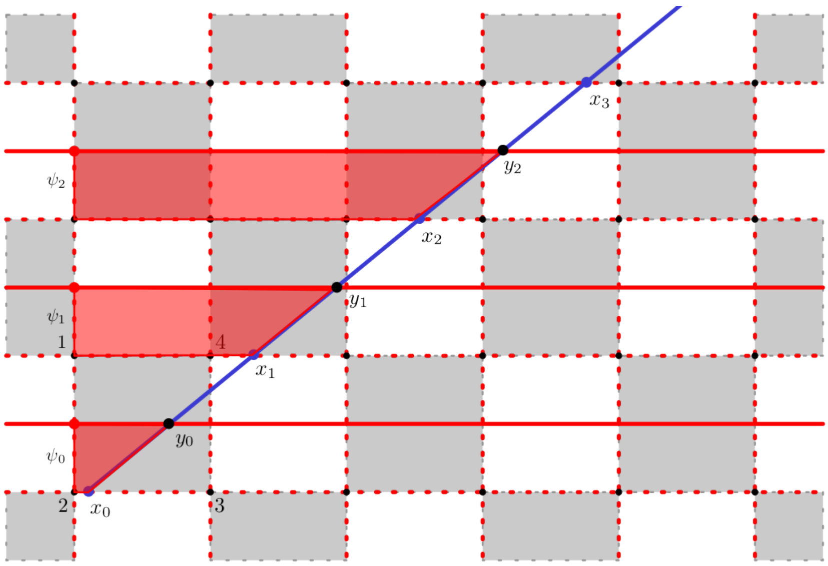


Figure 32: Half-picket domains

Remark 5.6 (Some symmetries)**.** Let $s := {}^p/_q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ and $Q_{p/q}$ oriented such that $Q_{p/q}(0)$ exists. Let P be the ordered matching of $Q_{p/q}$. First, mind that

$$G_k(s) = \Delta_{k+1}(s) + \operatorname{sgn}(s) \cdot (\rho_{k+1} + 2\Gamma_{k+1}(s)). \tag{$*$}$$

We check this easily by computing

$$\begin{aligned}
&(\Delta_{k+1} + \operatorname{sgn}(s) \cdot (\rho_{k+1} + 2\Gamma_{k+1}(s))) - G_k(s) \\
&= (\Delta_{k+1} + \operatorname{sgn}(s) \cdot (\rho_{k+1} + 2\Gamma_{k+1}(s))) - (\Delta_k(s) - \operatorname{sgn}(s) \cdot (\rho_k + 2\Gamma_k(s))) \\
&= -\operatorname{sgn}(s) \cdot ((1,1,0,0) + 2\Gamma_{k+1} + 2\Gamma_k - (\rho_{k+1} + \rho_k) - (2\Gamma_{k+1}(s) + 2\Gamma_k(s))) \\
&= 0.
\end{aligned}$$

Next, we want to make a connection between $\mathrm{u}_P(G_0(s))$ and $\mathrm{u}_P(G_\nu(s))$ for $\nu := 2p-1$. It holds using $(*)$ that

$$\begin{aligned}
&G_\nu(s) - G_0(s) \\
&= \Delta_{\nu+1}(s) + \operatorname{sgn}(s) \cdot (\rho_{\nu+1} + 2\Gamma_{\nu+1}(s)) - (\Delta_0(s) - \operatorname{sgn}(s) \cdot (\rho_0 + 2\Gamma_0(s))) \\
&= \Delta_{2p}(s) + \operatorname{sgn}(s) \cdot (\rho_{2p} + 2\Gamma_{2p}(s)) - (\Delta_0(s) - \operatorname{sgn}(s) \cdot (\rho_0 + 2\Gamma_0(s))) \\
&= \Delta_{2p}(s) - \Delta_0(s) + \operatorname{sgn}(s) \cdot (\rho_{2p} + \rho_0 + \Gamma_{2p}(s) + \Gamma_0(s)).
\end{aligned}$$

From definition it is clear that

$$\mathrm{u}_P(\Delta_0(s)) = \mathrm{u}_P(\Gamma_0(s)) = 0.$$

Because $Q_{p/q}(0)$ exists, the matching P is so that

$$\mathrm{u}_P(\Gamma_{2p}) = 0$$

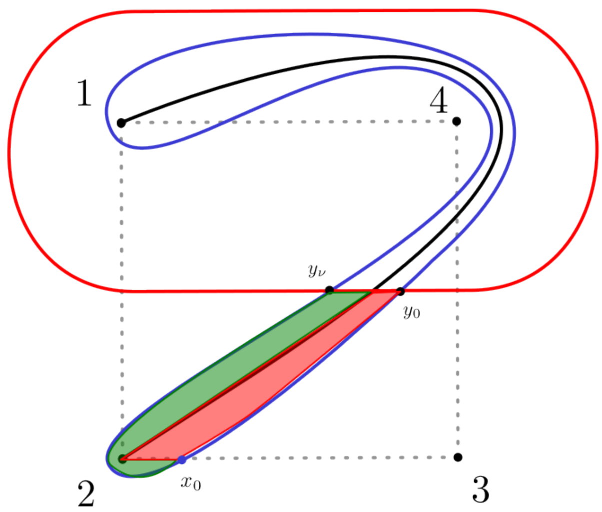


Figure 33: A convenient symmetry

because $\lambda_{2p}(s) = 2|q|$ is even. In proof 5.2 we see that $\mathrm{u}_P(\Delta_{2p}) = \mathrm{u}_P(\Delta_0)$, as both are Alexander gradings of different lifts of the same generator. Finally, note that

$$\rho_{2p} = \rho_0$$

as $2p$ and 0 are both even. With all that we get

$$\mathrm{u}_P(G_\nu(s)) - \mathrm{u}_P(G_0(s)) = \mathrm{sgn}(s)\,\mathrm{u}_P(2\rho_0).$$

We define

$$\lambda := \mathrm{sgn}(s)\,\mathrm{u}_P(2\rho_0) \in \{-1, 1\}.$$

Our goal is to show that

$$\mathrm{u}_P(G_{\nu-i}(s)) - \mathrm{u}_P(G_i(s)) = \lambda$$

holds for all $i \in \{0, \dots, p-1\}$. There is a nice geometric reasoning behind this (here we allude to results in proof 5.7): Remember remark 4.29, where we showed that we can determine $\mathrm{HFT}(Q_s)$ by isotoping an open component α of Q_s to the tangle border an take the border of a small closed neighbourhood of α. The resulting representative if $\mathrm{HFT}(Q_s)$ has a symmetry property around the arc α (exemplified in figure 33). We see that the points y_0 and y_ν have a certain Alexander grading difference of ± 1. Yet, because of the symmetry of $\mathrm{HFT}(Q_s)$ around α, this difference then stays constant between all generators $y_{\nu-i}$ and y_i for $i \in \{0, \dots, p-1\}$. We picked our representative of γ in the proof of proposition 5.2 in exactly that manner, hence the formulas should contain this symmetry. To achieve our goal, we show that

$$\mathrm{u}_P(G_{i+1}(s)) - \mathrm{u}_P(G_i(s)) = \mathrm{u}_P(G_{\nu-(i+1)}(s)) - \mathrm{u}_P(G_{\nu-i}(s)).$$

Let $\overline{\cdot} := \mathrm{u}_P(\,\cdot\,)$ and brace yourself. Using (∗) again we get in formulas

$$\begin{aligned}
&\overline{G}_{i+1}(s) - \overline{G}_i(s) \\
&= \overline{\Delta}_{i+1}(s) - \operatorname{sgn}(s)\cdot(\overline{\rho}_{i+1} + 2\overline{\Gamma}_{i+1}(s)) - (\overline{\Delta}_{i+1}(s) + \operatorname{sgn}(s)\cdot(\overline{\rho}_{i+1} + 2\overline{\Gamma}_{i+1}(s))) \\
&= -2\operatorname{sgn}(s)\cdot(\overline{\rho}_{i+1} + 2\overline{\Gamma}_{i+1}(s))
\end{aligned}$$

$$\begin{aligned}
&\overline{G}_{\nu-(i+1)}(s) - \overline{G}_{\nu-i}(s) \\
&= \overline{\Delta}_{\nu-i}(s) + \operatorname{sgn}(s)\cdot(\overline{\rho}_{\nu-i} + 2\overline{\Gamma}_{\nu-i}(s)) - (\overline{\Delta}_{\nu-i}(s) - \operatorname{sgn}(s)\cdot(\overline{\rho}_{\nu-i} + 2\overline{\Gamma}_{\nu-i}(s))) \\
&= 2\operatorname{sgn}(s)\cdot(\overline{\rho}_{\nu-i} + 2\overline{\Gamma}_{\nu-i}(s))
\end{aligned}$$

We therefore want to show that

$$\overline{\rho}_{\nu-i} + \overline{\rho}_{i+1} + 2\overline{\Gamma}_{\nu-i}(s)) + 2\overline{\Gamma}_{i+1}(s)) = 0 \tag{$**$}$$

First, notice that $i+1$ is even if and only if $2p-(i+1) = \nu - i$ is even. This implies that

$$\overline{\rho}_{i+1} = \overline{\rho}_{\nu-i}.$$

Hence

$$\overline{\rho}_{\nu-i} + \overline{\rho}_{i+1} = 2\overline{\rho}_{i+1} = \begin{cases} \varepsilon_1, & \text{if } i+1 \text{ is odd} \\ \varepsilon_2, & \text{if } i+1 \text{ is even} \end{cases} \tag{I}$$

where ε_1, ε_2 are like in definition 4.36. Furthermore, the existence of $Q_{p/q}(0)$ implies that

$$\overline{\Gamma}_y(s) = \begin{cases} \frac{1}{2}\varepsilon_1(\lfloor\frac{\lambda_y(s)}{2}\rfloor - \lceil\frac{\lambda_y(s)}{2}\rceil), & \text{if } y \text{ is odd,} \\ \frac{1}{2}\varepsilon_2(\lfloor\frac{\lambda_y(s)}{2}\rfloor - \lceil\frac{\lambda_y(s)}{2}\rceil), & \text{if } y \text{ is even.} \end{cases}$$

We can reformulate this to

$$\overline{\Gamma}_y(s) = \begin{cases} 0, & \text{if } \lambda_y(s) \text{ is even,} \\ -\frac{1}{2}\varepsilon_1, & \text{if } \lambda_y(s) \text{ is odd and } y \text{ is odd,} \\ -\frac{1}{2}\varepsilon_2, & \text{if } \lambda_y(s) \text{ is odd and } y \text{ is even.} \end{cases}$$

The notice from above gives us that

$$\overline{\Gamma}_{i+1}(s) = \overline{\Gamma}_{\nu-i}(s)$$

if and only if

$$\lambda_{i+1}(s) \equiv \lambda_{\nu-i}(s) \pmod 2.$$

We check

$$\begin{aligned}
\lfloor(\nu-i)|q/p|\rfloor &= \lfloor(2p-1-i)|q/p|\rfloor \\
&= 2|q| + \lfloor -(i+1)|q/p|\rfloor
\end{aligned}$$

Because $i \in \{0, \dots, p-1\}$ we know that $\lfloor -(i+1)|q/p| \rfloor$ is even if and only if $\lfloor (i+1)|q/p| \rfloor$ is odd, thus

$$\lambda_{i+1}(s) \not\equiv \lambda_{\nu-i}(s) \pmod 2$$

for all i and hence also

$$\overline{\Gamma}_{i+1}(s) \neq \overline{\Gamma}_{\nu-i}(s).$$

This implies that for all i holds

$$2(\overline{\Gamma}_{\nu-i}(s)) + \overline{\Gamma}_{i+1}(s))) = \begin{cases} -\varepsilon_1, & \text{if } i+1 \text{ is odd} \\ -\varepsilon_2, & \text{if } i+1 \text{ is even.} \end{cases} \tag{II}$$

If we put (I) and (II) into $(**)$, we have reached the goal.

Proposition 5.7. *Let $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ and $Q_{p/q}$ be oriented such that $Q_{p/q}(0)$ exists. Let P be the ordered matching of $Q_{p/q}$. Define the Alexander graded vector space*

$$G_P(p/q) \coloneqq \mathbb{F}_2\langle\{\mathrm{u}_P(G_k(p/q)) \mid k \in \{0, \dots, p-1\}\}\rangle$$

then

$$\mathrm{HF}(\mathrm{HFT}(Q_0), \mathrm{HFT}(Q_{p/q})) \cong \delta^0 t^{1/2} G_P(p/q) \otimes V$$

Proof. Let $\gamma \coloneqq \mathrm{HFT}(Q_{p/q})$. Let $\gamma_0 \coloneqq \mathrm{HFT}(Q_0)$ be symmetrically Alexander graded. Homotope γ such that it satisfies the same conditions as in the beginning of proof 5.2. Furthermore, homotope γ_0 such that $\eta^{-1}(\gamma_0)$ consists of parallel straight lines of slope zero and such that γ and γ_0 intersect minimally (possible by proof 4.34). We compute the Lagrangian Floer homology

$$\mathrm{HF}(\gamma_0, \gamma)$$

in the covering space using results from [LMZ21, Section 3.6]. As $p/q \neq 0$ we know we do this by looking at these minimal intersections of γ and γ_0. First, numerate the intersections $\tilde{\gamma} \cap \eta^{-1}(\gamma_0)$ along $\tilde{\gamma}$ starting from $x_0 \coloneqq \tilde{\gamma}(0)$ by y_i for $i \in \{0, \dots, 2p-1\}$.

Let ψ_i be the domain in covering space bordered by the base line, γ, γ_0 and the (punctured) horizontal line on which x_i lies (x_i is defined as in proof 5.2). We call this domains *half-picket domains* (see figure 32). By construction we have now that

$$\hat{A}(y_i) = \hat{A}(x_i) - \hat{A}(\psi_i).$$

Mind that γ_0 has no contribution because of the symmetric Alexander grading (Figure 28). The Alexander grading of our half-picket domains are given by

$$\hat{A}(\psi_i) = \mathrm{sgn}(p/q) \cdot (\rho_k + 2\Gamma_k(p/q)).$$

The already showed in proof 5.2 that

$$\hat{A}(x_i) = \Delta_i(p/q)$$

hence

$$\hat{A}(y_i) = G_k(p/q)$$

by definition. Hence we have (using lemma 4.34) that

$$\mathrm{HF}_*(\gamma_0,\, \gamma) \cong \mathbb{F}_2\langle\{\mathrm{u}_P(G_k({}^p\!/_q)) \mid k \in \{0,\, \ldots,\, 2p-1\}\}\rangle.$$

We showed in remark 5.6 that for all $i \in \{0, \ldots, p-1\}$ holds that

$$\mathrm{u}_P(G_i({}^p\!/_q)) - \mathrm{u}_P(G_{2p-1-i}({}^p\!/_q)) = \lambda$$

for a fixed $\lambda \in \{-1, 1\}$. Therefore we directly get

$$\mathbb{F}_2\langle\{\mathrm{u}_P(G_k({}^p\!/_q)) \mid k \in \{0,\, \ldots,\, 2p-1\}\}\rangle \cong t^{1/2} G_P({}^p\!/_q) \otimes V_*$$

and thereby the claim. We use that the gluing theorem 4.50, remark 4.17 and $Q_{p/q}(0) \neq U_2$ imply that our Lagragian Floer homology is δ-thin. □

Theorem 5.8. *Let L be a oriented rational link with fraction ${}^p\!/_q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ resulting in an ordered matching P. Let $G_P({}^p\!/_q)$ be as in the proposition 5.7, then*

$$\widehat{\mathrm{HFK}}(L) \cong \delta^0 t^{1/2} G_P({}^p\!/_q)$$

if L is a knot and

$$\widehat{\mathrm{HFK}}(L) \cong \delta^0 t^{1/2} G_P({}^p\!/_q) \otimes V$$

otherwise.

Proof. Follows from 5.7 and the gluing theorem 4.50. □

5.2 Derivation of a known formula

The following formula 5.12 for the Alexander polynomial of two-bridge links can be found in [Hos19, Theorem 1], but originally appearing in [Har79] and [Min82]. As often done, the authors fix a certain normalization for the fraction of rational links. We explain this normalization in remark 5.9, but mind that we normalize such that rational links are numerators not denominators, which explains the difference to [Hos19].

Remark 5.9 (Normalization)**.** Let L be a non-trivial rational link with fraction ${}^p\!/_q \in \mathbb{Q}\mathrm{P}^1$. As L is not the unlink ${}^p\!/_q \neq 0$ (corollary 2.47, we we can assume by Theorem 2.46 that $0 \leq q < p$. If $q = 0$ in this case, L would be the the unknot. Therefore we can assume that

$$-p < q < p.$$

and q is odd. In the following we are only interested in computing the Alexander polynomial of L. However, since this is mirror invariant, we can assume that

$$0 < q < p$$

as

$$\mathrm{m}(Q_{p/q}) = Q_{-p/q}.$$

In this case we can orient our link

$$L = Q_0 \cup Q_{p/q}$$

such that for $Q_{p/q}$ the tangle ends $\{1,\, 2\}$ are pointing *outwards* and $\{3, 4\}$ *inwards.*

Remark 5.10. We first want to make some observations about this particular normalization. Let $L = Q_0 \cup Q_{p/q}$ as in remark 5.9 with ordered matching P. As $p/q > 0$ we get for $y \in \mathbb{N}$

$$\lambda_y(p/q) = \lfloor y\frac{q}{p} \rfloor$$

and furthermore for the matching P that

$$\mathrm{u}_P(\Gamma_y(p/q)) = \frac{1}{2}(\lceil \frac{\lambda_y(p/q)}{2} \rceil - \lfloor \frac{\lambda_y(p/q)}{2} \rfloor) = \begin{cases} 0, & \text{if } \lambda_y(p/q) \text{ is even,} \\ 1/2, & \text{if } \lambda_y(p/q) \text{ is odd.} \end{cases}$$

as well as

$$\begin{aligned} \mathrm{u}_P(\Delta_k(p/q)) &= \mathrm{u}_P\left(-\mathrm{sgn}(p/q) \cdot \left(k \cdot (1,1,0,0) + 2 \cdot \sum_{y=1}^{k} (\Gamma_y(p/q) + \Gamma_{y-1}(p/q))\right)\right) \\ &= k - 2\sum_{y=1}^{k} \left(\mathrm{u}_P(\Gamma_y(p/q)) + \mathrm{u}_P(\Gamma_{y-1}(p/q))\right). \end{aligned}$$

This implies

$$\begin{aligned} \mathrm{u}_P(G_k(p/q)) &= \mathrm{u}_P(\Delta_k(p/q)) - \mathrm{u}_P(\rho_k) - \mathrm{u}_P(2\Gamma_k(p/q)) \\ &= \mathrm{u}_P(\Delta_k(p/q)) + \frac{1}{2} - \mathrm{u}_P(2\Gamma_k(p/q)). \end{aligned}$$

Lemma 5.11. *Let $L = Q_0 \cup Q_{p/q}$ be an oriented rational link as in remark 5.9 resulting in an ordered matching P. Then*

$$\mathrm{u}_P(G_k(p/q)) + \frac{1}{2} = \sum_{i=0}^{k} \varepsilon_i$$

where $\varepsilon_i := (-1)^{\lfloor iq/p \rfloor}$ for $k \in \mathbb{N}$.

Proof. In this proof the notate $\overline{\cdot} := \mathrm{u}_P(\,\cdot\,)$. First we observe that

$$\overline{G_0(p/q)} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 = (-1)^0 = \varepsilon_0.$$

Minding remark 5.10 we get for $i \in \mathbb{N}_{>0}$ that

$$\begin{aligned} &\overline{G_i(p/q)} - \overline{G_{i-1}(p/q)} = \overline{\Delta_i(p/q)} + \frac{1}{2} - \overline{2\Gamma_i(p/q)} - \overline{\Delta_{i-1}(p/q)} - \frac{1}{2} + \overline{2\Gamma_{i-1}(p/q)} \\ &= i - 2\sum_{y=1}^{i} \left(\overline{\Gamma_y(p/q)} + \overline{\Gamma_{y-1}(p/q)}\right) - \overline{2\Gamma_i(p/q)} - (i-1) + 2\sum_{y=1}^{i-1} \left(\overline{\Gamma_y(p/q)} + \overline{\Gamma_{y-1}(p/q)}\right) + \overline{2\Gamma_{i-1}(p/q)} \\ &= 1 - 4\overline{\Gamma_i(p/q)}. \end{aligned}$$

Hence

$$\overline{G_i(p/q)} - \overline{G_{i-1}(p/q)} = \begin{cases} 1, & \text{if } \lambda_i(p/q) \text{ is even,} \\ -1, & \text{if } \lambda_i(p/q) \text{ is odd,} \end{cases}$$

or in other words

$$\overline{G_i(p/q)} - \overline{G_{i-1}(p/q)} = (-1)^{\lfloor iq/p \rfloor} = \varepsilon_i.$$

□

In the following theorems the symbol $\doteq$ means equality up to multiplication by a unit of $\mathbb{Z}[t^{\pm 1}]$.

Theorem 5.12 (Hartley, Minkus)**.** *Let $L = Q_0 \cup Q_{p/q}$ be an oriented rational link as in remark 5.9. Then*

$$\Delta_L(t) \doteq \sum_{k=0}^{p-1} (-1)^k t^{\sum_{i=0}^{k} \varepsilon_i}$$

where $\varepsilon_i \coloneqq (-1)^{\lfloor iq/p \rfloor}$.

Proof. By proposition 4.5 we know that

$$\Delta_L(t) \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\mu(L)-1} \doteq \chi_{gr}(\widehat{\mathrm{HFK}}(L)).$$

We can use theorem 5.8 and get

$$\chi_{gr} \widehat{\mathrm{HFK}}(L) \doteq \chi_{gr}(\delta^0 t^{1/2} G_P(p/q))$$

if L is a knot and

$$\chi_{gr} \widehat{\mathrm{HFK}}(L) \doteq \chi_{gr}(\delta^0 t^{1/2} G_P(p/q) \otimes V) = \chi_{gr}(\delta^0 t^{1/2} G_P(p/q)) \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

otherwise (mind that the graded Euler characteristics are with respect to the Maslov and Alexander grading). Hence we know

$$\Delta_L(t) \doteq \chi_{gr}(\delta^0 t^{1/2} G_P(p/q)).$$

From definition 4.2 we get

$$\chi_{gr}(\delta^0 t^{1/2} G_P(p/q)) = \sum_{h,A\in\mathbb{Z}} (-1)^h t^A \dim(\delta^0 t^{1/2} G_P(p/q))_{h,A} = \dots$$

Mind that the δ-grading $0 = \delta = A - h$ implies that $\dim(\delta^0 G_P(p/q))_{h,A} = 0$ for all $h \neq A$.

$$\dots = \sum_{A\in\mathbb{Z}} (-1)^A t^A \dim(\delta^0 t^{1/2} G_P(p/q))_{A,A} = \dots$$

We now sum over the generators and not over gradings read of the Alexander gradings from proposition 5.7, which gives us

$$\dots = \sum_{k=0}^{p-1} (-1)^{\mathrm{u}_P(G_k(p/q))+1/2} t^{\mathrm{u}_P(G_k(p/q))+1/2} = \dots$$

for the ordered matching P of $Q_{p/q}$. We now apply the previous lemma 5.11 and get

$$\dots = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k} \varepsilon_i} t^{\sum_{i=0}^{k} \varepsilon_i} = \dots$$

where $\varepsilon_i := (-1)^{\lfloor iq/p \rfloor}$. The only observation left to make is that

$$(-1)^{\sum_{i=0}^{k} \varepsilon_i} = (-1)^{k+1}$$

for all $k \in \mathbb{N}$, as the sum changes parity for each additional summand and $\varepsilon_0 = 1$. Hence

$$\dots = \sum_{k=0}^{p-1} (-1)^{k+1} t^{\sum_{i=0}^{k} \varepsilon_i} \doteq \sum_{k=0}^{p-1} (-1)^{k} t^{\sum_{i=0}^{k} \varepsilon_i}$$

by multiplying with the unit $-1 \in \mathbb{Z}[t^{\pm 1}]$. □

Hoste additionally showed the following formula for the multivariable Alexander polynomial.

Theorem 5.13 (Hoste, [Hos19, Theorem 2]). *Let L be a oriented rational 2-component link as in remark 5.9. Then*

$$\Delta_L(x, y) \doteq \sum_{i=1}^{p/2} \varepsilon_{2i-1} x^{\sum_{j=1}^{i-1} \varepsilon_{2j}} y^{\frac{\varepsilon_{2i-1}-1}{2} + \sum_{k=1}^{i-1} \varepsilon_{2k-1}}$$

where $\varepsilon_i := (-1)^{\lfloor iq/p \rfloor}$.

The author assumes that this formula can be proven in a similar manner as the above one by using the multi-graded link Floer homology $\widehat{\mathrm{HFL}}$ and a Alexander bigrading (in addition to the δ-grading) for the occurring immersed curves. This suspicion is based on the fact that we know that the gluing theorem [Zib20, Theorem 5.9] holds more generally for link Floer homology and that the formulas from definition 5.1 stay the same.

5.3 Connection with even continued fractions

To draw a connection to the earlier result for rational links (3.17), we first need to use our knowledge (4.42) about how rational curves behave under the generators $\tau_1, \tau_2 \in \mathrm{Mod}(S^2_4)$ (2.18). This will also lead to a (maybe already known) statement (5.20) concerning the number of generators in the highest Alexander degree of the knot Floer homology of rational links.

Definition 5.14. Let T be a oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$ absolutely Alexander graded. For $s \in \{a, b, c, d\}$ we define

$$M_s(\gamma) := \max_{x \in X_s(\gamma)} A(x)$$

and

$$m_s(\gamma) := \min_{x \in X_s(\gamma)} A(x).$$

Corollary 5.15. *Let Q be a oriented rational tangle and $\gamma \coloneqq \mathrm{HFT}(Q)$ absolutely Alexander graded. If $Q(0)$ exists, then*

$$M_-(\gamma) \coloneqq M_b(\gamma) = M_d(\gamma) \quad \text{and} \quad m_-(\gamma) \coloneqq m_b(\gamma) = m_d(\gamma).$$

If $Q(\infty)$ exists, then

$$M_|(\gamma) \coloneqq M_a(\gamma) = M_c(\gamma) \quad \text{and} \quad m_|(\gamma) \coloneqq m_a(\gamma) = m_c(\gamma)$$

Proof. Follows from corollary 4.63. □

Lemma 5.16. *Let Q be a cyclically oriented rational tangle, $\gamma \coloneqq \mathrm{HFT}(Q)$ absolutely Alexander graded and let $a \in 2\mathbb{Z} \setminus \{0\}$. Then*

$$\begin{aligned} M_|(\gamma) \geq M_-(\gamma) &\implies M_-(\tau_1^a\gamma) = M_|(\gamma) + 1/2 \\ m_|(\gamma) \leq m_-(\gamma) &\implies m_-(\tau_1^a\gamma) = m_|(\gamma) - 1/2 \end{aligned}$$

and

$$\begin{aligned} M_-(\gamma) \geq M_|(\gamma) &\implies M_|(\tau_2^a\gamma) = M_-(\gamma) + 1/2 \\ m_-(\gamma) \leq m_|(\gamma) &\implies m_|(\tau_2^a\gamma) = m_-(\gamma) - 1/2. \end{aligned}$$

Proof. Let $M_|(\gamma) \geq M_-(\gamma)$ and $x \in X_c(\gamma)$ with $A(x) = M_|(\gamma)$. By remark 4.42 and because a is even, we know that x induces at least one provisional generator $x_1 \in X_d(\tau_1^a\gamma)$ with

$$A(x_1) = A(x) + 1/2.$$

Hence $A(x_1) > M_-(\gamma)$, which means that x_1 cannot cancel itself with any existing generator and therefore does not cancel at all (Remark 4.42).

The other cases work similarly. □

We now have the tools to give an alternative proof of proposition 3.17.

Proof of proposition 3.17. Remember, L is a rational with cyclical even continued fraction $[a_1, \ldots, a_n]$, where we exclude the trivial case $L = U_2$ with even continued fraction $[0]$ for which the formulas hold. By the definitions 2.35 and 3.15 this means

$$L = Q_0 \cup Q_{p/q}$$

where $Q_{p/q}$ is cyclically oriented and $p/q = [a_1, \ldots, a_n]$. Furthermore Conway's algorithm 2.25 gives that

$$Q_{p/q} = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

Mind that Q_∞ and Q_0 have the same cyclical orientation as $Q_{p/q}$. Let $\gamma_\infty \coloneqq \mathrm{HFT}(Q_\infty)$ and $\gamma_0 \coloneqq \mathrm{HFT}(Q_0)$ both be symmetrically Alexander graded (see figure 28). Then

$$M_-(\gamma_\infty) = m_-(\gamma_\infty) = 0 \quad \text{and} \quad M_|(\gamma_\infty) = -\infty, \quad m_|(\gamma_\infty) = \infty.$$

As well as

$$M_|(\gamma_0) = m_|(\gamma_0) = 0 \quad \text{and} \quad M_-(\gamma_0) = -\infty, \quad m_-(\gamma_0) = \infty.$$

Now, mind that theorem 4.27 shows for $\gamma \coloneqq \mathrm{HFT}(Q_{p/q})$ that

$$\gamma = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\gamma_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\gamma_0, & \text{if } n \text{ is odd.} \end{cases}$$

where we fix an (absolute) Alexander grading on γ through this equality and remark 4.42. Note that proposition 4.69 implies that γ is symmetrically Alexander graded.

If n is even, applying lemma 5.16 successively to

$$\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\gamma_\infty$$

yields that

$$M_-(\gamma) = \frac{n}{2}$$

if $a_1 \neq 0$ or $n \leq 1$ and

$$M_-(\gamma) = \frac{n-2}{2}$$

otherwise.

If n is odd, applying lemma 5.16 successively to

$$\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\gamma_0$$

yields equally that

$$M_-(\gamma) = \frac{n}{2}$$

if $a_1 \neq 0$ and

$$M_-(\gamma) = \frac{n-2}{2}$$

otherwise.

By proposition 4.70

$$\mathrm{HF}(\gamma_0, \gamma)$$

is symmetrically Alexander graded, hence lemma 4.57 implies that the relative isomorphism of the gluing theorem 4.50 induces an absolute isomorphism

$$\widehat{\mathrm{HFK}}_*(L) \otimes V_*^{\otimes(2-\mu(L))} = \mathrm{HF}_*(\gamma_0, \gamma).$$

Remember that L can at most have two components. Furthermore, lemma 4.62 shows that

$$\mathrm{HF}_*(\gamma_0, \gamma) = V_-(\gamma) \otimes V_*.$$

We use this equality to get

$$\begin{aligned}\max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L,A) \neq 0\} &= \max\{A \in \mathbb{Z} \mid (\mathrm{HF}_*(\gamma(0),\gamma))_A \neq 0\} - \frac{2-\mu(L)}{2} \\ &= \max\{A \in \mathbb{Z} \mid (V_-(\gamma) \otimes V_*)_A \neq 0\} - \frac{2-\mu(L)}{2} \\ &= \max\{A \in \mathbb{Z} \mid (V_-(\gamma))_A \neq 0\} + \frac{1}{2} - \frac{2-\mu(L)}{2}\end{aligned}$$

and thus

$$\begin{aligned} g(L) &\overset{4.6}{=} \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L,A) \neq 0\} - \mu(L) + 1 \\ &= \max\{A \in \mathbb{Z} \mid (V_-)_A \neq 0\} + \frac{1}{2} - \frac{2-\mu(L)}{2} - \mu(L) + 1 \\ &= M_-(\gamma) + \frac{1-\mu(L)}{2}.\end{aligned}$$

Mind now that L is a knot if n even and a two-component link, if n is odd (Remark 3.14). Finally, we get

$$\begin{aligned} g(L) &= M_-(\gamma) + \frac{1-\mu(L)}{2}. \\ &= \frac{n}{2} + \begin{Bmatrix} 0, & \text{if } n \text{ is even} \\ -\frac{1}{2}, & \text{if } n \text{ is odd} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{Bmatrix}\end{aligned}$$

for $a_1 \neq 0$ or $n \leq 1$ and

$$\begin{aligned} g(L) &= M_-(\gamma) + \frac{1-\mu(L)}{2}. \\ &= \frac{n-2}{2} + \begin{Bmatrix} 0, & \text{if } n \text{ is even} \\ -\frac{1}{2}, & \text{if } n \text{ is odd} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{Bmatrix} - 1\end{aligned}$$

if $a_1 = 0$ and $n > 1$. □

Definition 5.17. Let T be a oriented Conway tangle and $\gamma := \mathrm{HFT}(T)$ absolutely Alexander graded. For $s \in \{a, b, c, d\}$ we define

$$L_s(\gamma) := |\{x \in X_s(\gamma) \mid A(x) = M_s(\gamma)\}|$$

and

$$l_s(\gamma) := |\{x \in X_s(\gamma) \mid A(x) = m_s(\gamma)\}|.$$

Corollary 5.18. *Let Q be a oriented rational tangle and $\gamma \coloneqq \mathrm{HFT}(Q)$ absolutely Alexander graded. If $Q(0)$ exists, then*

$$L_-(\gamma) \coloneqq L_b(\gamma) = L_d(\gamma) \quad \text{and} \quad l_-(\gamma) \coloneqq l_b(\gamma) = l_d(\gamma).$$

If $Q(\infty)$ exists, then

$$L_|(\gamma) \coloneqq L_a(\gamma) = L_c(\gamma) \quad \text{and} \quad l_|(\gamma) \coloneqq l_a(\gamma) = l_c(\gamma)$$

Proof. Follows from corollaries 5.15 and 4.63. □

Lemma 5.19. *Let Q be a oriented rational tangle, $\gamma \coloneqq \mathrm{HFT}(Q)$ absolutely Alexander graded and let $a \in 2\mathbb{Z} \setminus \{0\}$. Then*

$$M_|(\gamma) \geq M_-(\gamma) \implies L_-(\tau_1^a\gamma) = \frac{|a|}{2} L_|(\gamma)$$
$$m_|(\gamma) \leq m_-(\gamma) \implies l_-(\tau_1^a\gamma) = \frac{|a|}{2} l_|(\gamma)$$

and

$$M_-(\gamma) \geq M_|(\gamma) \implies L_|(\tau_2^a\gamma) = \frac{|a|}{2} L_-(\gamma)$$
$$m_-(\gamma) \leq m_|(\gamma) \implies l_|(\tau_2^a\gamma) = \frac{|a|}{2} l_-(\gamma).$$

Proof. Let $M_|(\gamma) \geq M_-(\gamma)$. From lemma 5.16 and remark 4.42 we get that every $x \in X_c(\gamma)$ with $A(x) = M_|(\gamma)$ induces exactly $|a|/2$ provisional generators $x_i \in X_d(\tau_1^a\gamma)$ with

$$A(x_i) = A(x) + 1/2 = M_-(\tau_1^a\gamma).$$

As $A(x_i) > M_-(\gamma)$ and the x_i's cannot cancel themselves with any existing generators and therefore do not cancel at all (Remark 4.42).

The other cases work similarly. □

Proposition 5.20. *Let L be a rational link with cyclical even continued fraction $[a_1, \ldots, a_n]$. Then for*

$$A_{\max} \coloneqq \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(L, A) \neq 0\}$$

holds that

$$\dim \widehat{\mathrm{HFK}}_*(L, A_{\max}) = \begin{cases} 2, & \text{if } n = 1 \wedge a_1 = 0, \\ \prod_{i=3}^{n} |a_i|/2, & \text{if } n > 1 \wedge a_1 = 0, \\ \prod_{i=1}^{n} |a_i|/2, & \text{if } n = 0 \vee a_1 \neq 0. \end{cases}$$

Proof. By the definitions 2.35 and 3.15

$$L = Q_0 \cup Q_{p/q}$$

where $Q_{p/q}$ is cyclically oriented and $p/q = [a_1, \ldots, a_n]$. If $n = 1$ and $a_1 = 0$, then $L = Q_0 \cup Q_0 = U_2$ and hence $\dim \widehat{\mathrm{HFK}}_*(L, A_{\max}) = \dim \widehat{\mathrm{HFK}}_*(U_2, 0) = 2$. Therefore exclude this case. Furthermore, Conway's algorithm 2.25 gives that

$$Q_{p/q} = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}Q_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}Q_0, & \text{if } n \text{ is odd.} \end{cases}$$

Mind that Q_∞ and Q_0 have the same cyclical orientation as $Q_{p/q}$. Let $\gamma_\infty \coloneqq \mathrm{HFT}(Q_\infty)$ and $\gamma_0 \coloneqq \mathrm{HFT}(Q_0)$ both be symmetrically Alexander graded (see figure 28).

Now, mind that theorem 4.27 shows for $\gamma \coloneqq \mathrm{HFT}(Q_{p/q})$ that

$$\gamma = \begin{cases} \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\gamma_\infty, & \text{if } n \text{ is even,} \\ \tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\gamma_0, & \text{if } n \text{ is odd.} \end{cases}$$

where we set an (absolute) Alexander grading on γ through this equality. Note that lemma 4.66 implies that γ is symmetrically Alexander graded.

The idea of this proof is quite similar to the idea of the last proof, but instead of just keeping track of the maximal Alexander grading per site, we keep track of the number of generators having this maximal Alexander grading. From figure 28 wee see that

$$L_-(\gamma_\infty) = l_-(\gamma_\infty) = 1 \quad \text{and} \quad L_|(\gamma_\infty) = l_|(\gamma_\infty) = 0.$$

As well as

$$L_|(\gamma_0) = l_|(\gamma_0) = 1 \quad \text{and} \quad L_-(\gamma_0) = l_-(\gamma_0) = 0.$$

If n is even, applying lemma 5.19 successively to

$$\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_2^{a_n}\gamma_\infty$$

yields that

$$L_-(\gamma) = \prod_{i=1}^{n} |a_i|/2$$

if $a_1 \neq 0$ or $n \leq 1$ and

$$L_-(\gamma) = \prod_{i=3}^{n} |a_i|/2$$

otherwise.

If n is odd, applying lemma 5.16 successively to

$$\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_1^{a_n}\gamma_0$$

yields equally that

$$L_-(\gamma) = \prod_{i=1}^{n} |a_i|/2$$

if $a_1 \neq 0$ and

$$L_-(\gamma) = \prod_{i=3}^{n} |a_i|/2$$

otherwise.

By proposition 4.70

$$\mathrm{HF}(\gamma_0, \gamma)$$

is symmetrically Alexander graded, hence lemma 4.57 implies that the relative isomorphism of the gluing theorem 4.50 induces an absolute isomorphism

$$\widehat{\mathrm{HFK}}_*(L) \otimes V_*^{\otimes(2-\mu(L))} = \mathrm{HF}_*(\gamma_0, \gamma).$$

Furthermore, lemma 4.62 shows that

$$\mathrm{HF}_*(\gamma_0, \gamma) = V_-(\gamma) \otimes V_*.$$

Let $n(V)$ of an Alexander graded vector space be the number of generators in the highest supported Alexander grade. Minding that

$$n(V) = n(V \otimes V_*)$$

we get

$$\begin{aligned}\dim \widehat{\mathrm{HFK}}_*(L,\, A_{\max}) &= n(\widehat{\mathrm{HFK}}_*(L)) = n(\widehat{\mathrm{HFK}}_*(L) \otimes V_*^{\otimes(2-\mu(L))}) \\ &= n(\mathrm{HF}_*(\gamma_0, \gamma)) = n(V_-(\gamma) \otimes V_*) \\ &= n(V_-(\gamma)) \\ &= L_-\gamma\end{aligned}$$

and hence the claim. □

Corollary 5.21. *Let L be a rational link in cyclical orientation with even continued fraction $[a_1, \ldots, a_n]$. Then*

$$L \textit{ is fibred} \iff \forall_{i \in \{1,\ldots,n\}} : |a_i| = 2$$

if $a_1 \neq 0$ and

$$L \textit{ is fibred} \iff \forall_{i \in \{3,\ldots,n\}} : |a_i| = 2$$

otherwise.

Proof. By corollary 2.47 L is split only if $[a_1, \ldots, a_n] = [0]$ and from remark 4.9 we know that the two-component unlink is not fibred. The statement then follows from proposition 5.20 and theorem 4.11. □

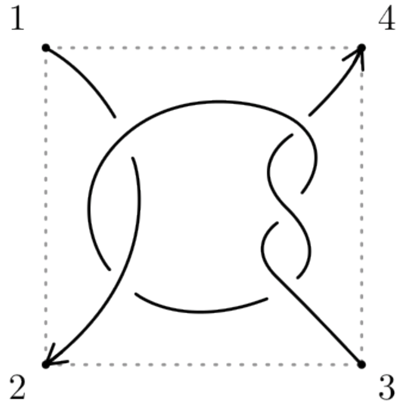


Figure 34: Oriented (2,-3)-pretzel tangle $T_{2,-3}$

6 Rational closures of the (2,-3)-pretzel tangle

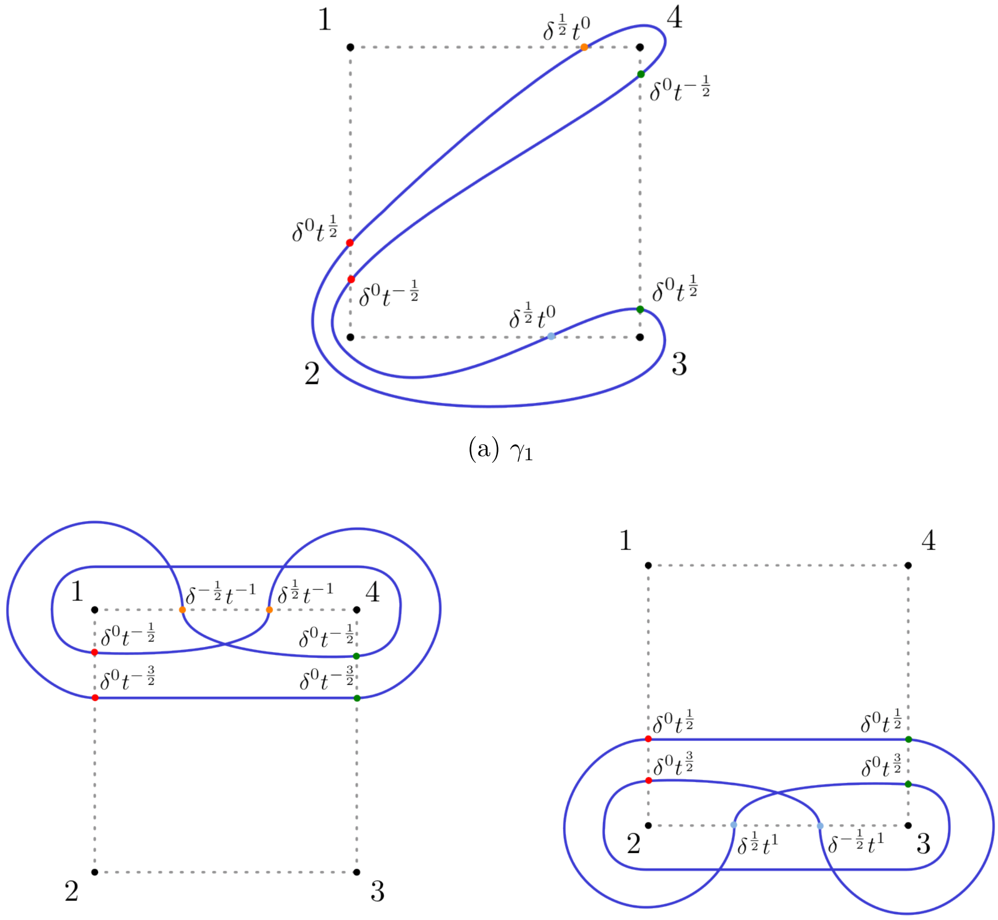
In this chapter, we are concerned about rational closures (3.20) of the (2,-3)-pretzel tangle $T_{2,-3}$ oriented like in figure 34. We first look at its multicurve tangle invariant which consists of one rational and two special components. Then we simplify the computation of the Lagrangian intersection Floer homology by interpreting intersections with special curves as intersections with rational curves plus stabilization. The main theorem then expresses the genus of these rational closures as the maximum of two genera of rational links.

6.1 Associated multicurve

Remark 6.1. From now on all $\mathfrak{r}({}^{p}\!/\!_{q})$ for ${}^{p}\!/\!_{q} \in \mathbb{Q}\mathrm{P}^1$ lying on a parametrized S^2_4 shall have symmetric Alexander grading (if existent) and an absolute δ-grading such that vertical generators have δ-degree zero (Remark 4.39). Furthermore, we fix the bigrading of $\mathfrak{s}_1(0; x, y)$ for $(x, y) \in \{(1, 4), (2, 3)\}$ for the depicted orientation as in figure 24.

Example 6.2 (cf. [Zib20, Example 2.26])**.** Examine the curves in figure 35. The multicurve $\mathrm{HFT}(T_{2,-3}) = \{\gamma_1, \gamma_2, \gamma_3\}$ consists of these three components. We can fix the bigrading such that the invariant consist of one embedded rational component $\gamma_1 = \mathfrak{r}({}^{1}\!/\!_{2})$ and two immersed special components $\gamma_2 = t^{-1}\,\mathfrak{s}_1(0; 1, 4)$ and $\gamma_3 = t^1\,\mathfrak{s}_1(0; 2, 3)$.

Remark 6.3. From this knowledge and theorem 4.28 we get a second (Remark 2.44) proof that $T_{2,-3}$ is not trivial. Furthermore, theorem 4.30 shows that $T_{2,-3}$ is not split as tangle. Mind that $\mathrm{HFT}(T_{2,-3})$ has the minimal non-zero number of special components, as the conjugation symmetry from theorem 4.26 shows that special components always occur in pairs.



(a) γ_1

(b) γ_2

(c) γ_3

Figure 35: The bigraded multicurve $\mathrm{HFT}(T_{2,-3})$

6.2 Reducing special to rational parings

Lemma 6.4. *Let* $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$*,* $p+q \equiv 1 \pmod 2$*. Let* $\mathfrak{r}(p/q)$*,* $\mathfrak{r}(0)$ *and* $\mathfrak{s}_1(0;x,y)$ *with gradings as in remark 6.1 lie on* $S^2_4(T)$ *for a cyclically oriented Conway tangle* T*. Then*

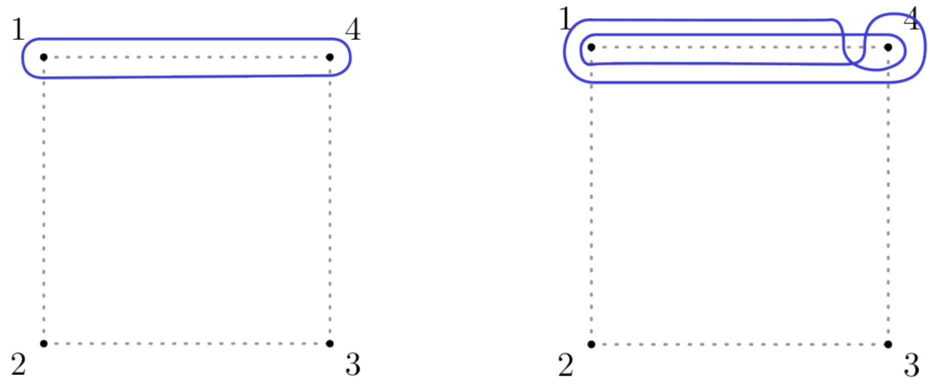
$$\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}_1(0;x,y)) \quad = \quad \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) \otimes V$$

for $(x,y) \in \{(1,4),(2,3)\}$*.*

Proof. We examine the case $(x,y) = (1,4)$, but the other case works analogously.

From remark 4.33 we know that we can homotope $\mathfrak{r}(0)$ and $\mathfrak{s}_1(0;1,4)$ to a small neighbourhood of the site d such that

$$|\mathfrak{r}(p/q) \cap \mathfrak{r}(0)| = 2|p|$$
$$|\mathfrak{r}(p/q) \cap \mathfrak{s}_1(0;1,4)| = 4|p|.$$



(a) $\mathfrak{r}(0)$ (b) $\mathfrak{s}_1(0;1,4)$

Figure 36: Convenient representatives

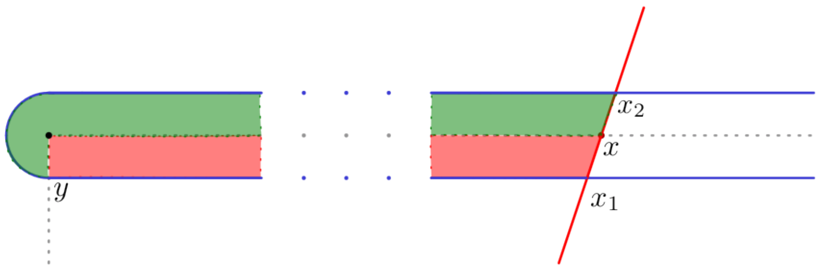


Figure 37: Computation of $\mathrm{HF}(\mathfrak{r}(p/q), \mathfrak{r}(0))$

Then lemma 4.34 shows that the curves intersect minimally. Furthermore, the two intersections of $\mathfrak{s}_1(0;1,4)$ with site d shall lie right of all intersections of $\mathfrak{r}(p/q)$ with site d. See figure 36 for an illustration. As $\mathfrak{r}(p/q)$ and $\mathfrak{r}(0)$ are not homotopic (by the assumption $p/q \neq 0$) we can compute the Lagrangian Floer homology from the intersection points. As $\mathfrak{r}(p/q)$ intersects the parametrization minimally by assumption all intersections $\mathfrak{r}(p/q) \cap \mathfrak{r}(0)$ look (up to homotopy) like in figure 37 ($\mathfrak{r}(p/q)$ is red, $\mathfrak{r}(0)$ is blue). For the same reason all intersections $\mathfrak{r}(p/q) \cap \mathfrak{s}_1(0;x,y)$ look (up to homotopy) like in figure 38 ($\mathfrak{r}(p/q)$ is red, $\mathfrak{s}_1(0;x,y)$ is blue).

If we calculate $\mathrm{HF}(\mathfrak{r}(p/q), \mathfrak{r}(0))$, we see that for every $x \in X_d(\mathfrak{r}(p/q))$ we get generators $x_1, x_2 \in \mathrm{HF}(\mathfrak{r}(p/q), \mathfrak{r}(0))$ as seen in figure 37. In fact, because of the minimality of the number of intersections and lemma 4.59 this induces a 1:2 correspondence between $X_d(\mathfrak{r}(p/q))$ and the generators of $\mathrm{HF}(\mathfrak{r}(p/q), \mathfrak{r}(0))$.

We can compute the bigradings of x_1 and x_2 relative to x by using the red and green basic connecting domains along the site d to the puncture 1 (Remark 4.43). This gives

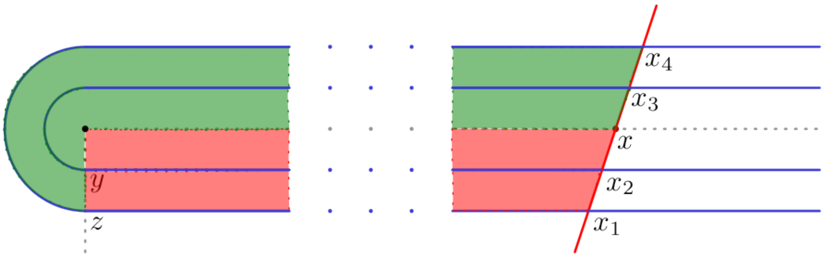


Figure 38: Computation of $\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}_1(0;1,4))$

us

$$\begin{aligned}\delta(x_1) &= \delta(x_2) = \delta(y) - \delta(x) + 1/2 = -\delta(x) + 1/2\\ A(x_1) &= A(y) - A(x) - 1/2 = -A(x) - 1/2\\ A(x_2) &= A(y) - A(x) + 1/2 = -A(x) + 1/2\end{aligned}$$

beause we assumed $\delta(y) = A(y) = 0$. Let W_d be the bigraded vector space $V_d(\mathfrak{r}(p/q))$ but with inversed Alexander and δ-gradings. We just showed that

$$\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) = W_d \otimes \delta^{1/2} V.$$

If we calculate $\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}_1(0;1,4))$, we see that for every $x \in X_d(\mathfrak{r}(p/q))$ we get generators $x_1, x_2, x_3, x_4 \in \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}_1(0;1,4))$ as seen in figure 38. In fact, because of the minimality of the number intersections and lemma 4.59 this induces a 1:4 correspondence between $X_d(\mathfrak{r}(p/q))$ and the generators of $\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}_1(0;1,4))$.

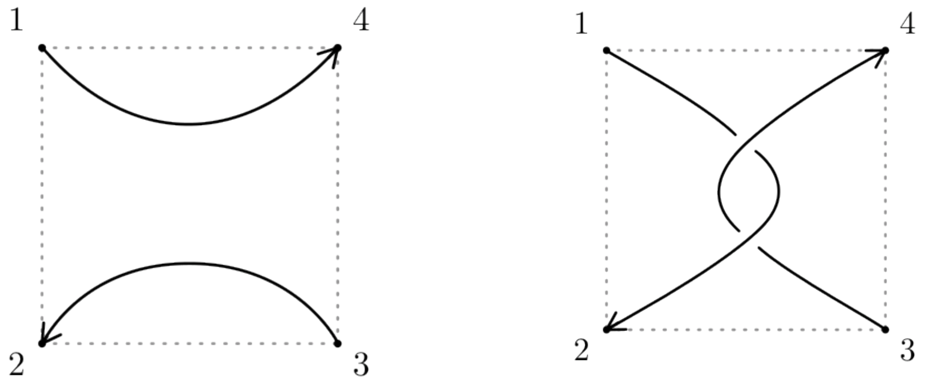
We can compute the bigradings of x_1, x_2, x_3 and x_4 relative to x by using the basic connecting domains along the curves to the puncture 1 depicted in figure 38. This gives us

$$\begin{aligned}\delta(x_1) &= \delta(x_4) = \delta(z) - \delta(x) + 1/2\\ \delta(x_2) &= \delta(x_3) = \delta(y) - \delta(x) + 1/2\\ A(x_1) &= A(z) - A(x) - 1/2 = -A(x) - 1\\ A(x_4) &= A(z) - A(x) + 1/2 = -A(x)\\ A(x_2) &= A(y) - A(x) - 1/2 = -A(x)\\ A(x_3) &= A(y) - A(x) + 1/2 = -A(x) + 1\end{aligned}$$

because $\delta(y) = \delta(z) = 0$, $A(y) = 1/2$ and $A(z) = -1/2$. Then it holds

$$\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) = W_d \otimes \delta^{1/2} V \otimes V.$$

□



(a) Q_0 (b) $Q_{1/2}$

Figure 39: Oriented rational tangles

6.3 Conclusions

This last section contains the final results of our work. We will see that the genus of the rational closure $T_{2,-3}(p/q)$ only depends on the genus of $Q_0(p/q)$ plus some constant and that $T_{2,-3}(p/q)$ is fibred if and only if $Q_0(p/q)$ is fibred. Using the results of section 3.3 and 5.3, we then deduce convenient methods to determine both properties.

Remark 6.5. One could ask, if these two properties (genus and fibredness) of a rational particular closure $T_{2,-3}(p/q)$ could be determined by computing its Alexander polynomial. This is possible if the link is δ-thin (see the end of section 4.1). In [KWZ21, Example 8.3] the authors showed (after some rotation) that $T_{2,-3}(p/q)$ is δ-thin if and only if

$$p/q \in (1/2,\, 0]$$

i.e. $p/q > 1/2$ (including ∞) or $p/q \leq 0$. So not only is it unclear whether we would get any convenient formulas (dependent on p/q) at all by using the Alexander polynomial, but it is even impossible for an infinite amount of closures. This also shows that $T_{2,-3}(p/q)$ cannot be a rational link for $p/q \notin (1/2,\, 0]$ (Remark 4.17).

Remark 6.6. In the following results we always exclude the case of the 0-closure of $T_{2,-3}$, but mind that $T_{2,-3}(0)$ is simply the unknot. Hence there is no actual benefit of adding the 0-closure to the formulas, which would make it more illegible.

Remark 6.7. In this section we fix the orientations of $Q_{1/2}$ and Q_0 to give the ordered matching $\{(1,4), (3,4)\}$ as illustrated in figure 39.

Observe that in the next propositions we only prove an isomorphism of unigraded vector spaces, concretely an isomorphism with respect to the Alexander grading.

Proposition 6.8. *Let $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ with p even and q odd. Then*

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) \quad = \quad \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \end{array}$$

holds.

Proof. As p is even and q odd, we know from the connectivities (Remark 2.33) that $T_{2,-3}(p/q)$ and $Q_{1/2}(p/q)$ are knots and $Q_0(p/q)$ is a two-component link, which we want to keep in mind when using the gluing theorem 4.50. Hence

$$\begin{aligned} & \widehat{\mathrm{HFK}}(T_{2,-3}(p/q)) \otimes V \\ \overset{4.50}{\cong}\; & \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}\, Q_{-p/q}),\, \mathrm{HFT}(T_{2,-3})) \\ \overset{6.2}{=}\; & \mathrm{HF}(\mathrm{HFT}(Q_{p/q}),\, \{\gamma_1, \gamma_2, \gamma_3\}) \\ \overset{4.45}{=}\; & \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_1) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_2) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_3) \\ \overset{6.2}{=}\; & \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, t^{-1}\, \mathfrak{s}(0;1,4)) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, t^{1}\, \mathfrak{s}(0;2,3)) \\ \overset{4.46}{=}\; & \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus t^{-1}\, \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}(0;1,4)) \oplus t^{1}\, \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}(0;2,3)) \\ \overset{6.4}{=}\; & \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus \Big(t^{-1}\, \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) \otimes V\Big) \oplus \Big(t^{1}\, \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) \otimes V\Big) \\ \overset{(*)}{\cong}\; & \Big(\widehat{\mathrm{HFK}}(Q_{1/2}(p/q)) \otimes V\Big) \oplus \Big(t^{-1}\, \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V\Big) \oplus \Big(t^{1}\, \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V\Big) \\ =\; & \Big(\widehat{\mathrm{HFK}}(Q_{1/2}(p/q)) \oplus t^{-1}\, \widehat{\mathrm{HFK}}(Q_0(p/q)) \oplus t^{1}\, \widehat{\mathrm{HFK}}(Q_0(p/q))\Big) \otimes V. \end{aligned}$$

Mind that the isomorphism $(*)$ does not have to respect the δ-grading, as the δ-shifts coming from the gluing theorem could be different. However, proposition 4.70 and lemma 4.57 show that each one of three relative isomorphism from the gluing theorem is absolute with respect to the Alexander grading, hence $(*)$ is an absolute isomorphism (of unigraded vector spaces) with respect to the Alexander grading. Altogether we have a relative isomorphism

$$\widehat{\mathrm{HFK}}_*(T_{2,-3}(p/q)) \cong \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \oplus t^{-1}\, \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \oplus t^{1}\, \widehat{\mathrm{HFK}}_*(Q_0(p/q))$$

of finite-dimensional symmetrically Alexander graded (4.55) vector spaces, therefore lemma 4.56 shows that this is an absolute isomorphism. □

Proposition 6.9. *Let $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ with p odd and q even. Then*

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) \quad = \quad \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2} \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2} \end{array}$$

holds.

Proof. This proof only differs from the proof of Proposition 6.8 in situations when the gluing theorem 4.50 is applied. As in this case p is odd and q even, we know from the connectivities (Remark 2.33) that $T_{2,-3}(p/q)$ and $Q_{1/2}(p/q)$ are two-component links and $Q_0(p/q)$ is a knot. Hence

$$\begin{aligned}
&\widehat{\mathrm{HFK}}(T_{2,-3}(p/q))\\
\overset{4.50}{\cong}\;& \mathrm{HF}(\mathrm{HFT}(\mathrm{mr}\, Q_{-p/q}),\, \mathrm{HFT}(T_{2,-3}))\\
\overset{6.2}{=}\;& \mathrm{HF}(\mathrm{HFT}(Q_{p/q}),\, \{\gamma_1,\gamma_2,\gamma_3\})\\
\overset{4.45}{=}\;& \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_1) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_2) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, \gamma_3)\\
\overset{6.2}{=}\;& \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, t^{-1}\,\mathfrak{s}(0;1,4)) \oplus \mathrm{HF}(\mathfrak{r}(p/q),\, t^{1}\,\mathfrak{s}(0;2,3))\\
\overset{4.46}{=}\;& \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus t^{-1}\,\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}(0;1,4)) \oplus t^{1}\,\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{s}(0;2,3))\\
\overset{6.4}{=}\;& \mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(1/2)) \oplus \left(t^{-1}\,\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) \otimes V\right) \oplus \left(t^{1}\,\mathrm{HF}(\mathfrak{r}(p/q),\, \mathfrak{r}(0)) \otimes V\right)\\
\overset{(*)}{\cong}\;& \widehat{\mathrm{HFK}}(Q_{1/2}(p/q)) \oplus \left(t^{-1}\,\widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2}\right) \oplus \left(t^{1}\,\widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2}\right)
\end{aligned}$$

which shows the claim using the same closing arguments as in proof 6.8. □

Theorem 6.10. *Let $p/q \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ with $p+q \equiv 1 \pmod 2$ and let*

$$c(p/q) = \begin{cases} 1, & \text{if } p \text{ is odd, } q \text{ is even} \\ 2, & \text{if } p \text{ is even, } q \text{ is odd.} \end{cases}$$

Then we have

$$g\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \max\left\{\begin{array}{c} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + c(p/q) \end{array}\right\}.$$

Proof. Case I: p is even and q is odd.
In this case proposition 6.8 gives us

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)). \end{array}$$

We use theorem 4.6 to deduce (mind that $T(p/q)$ is a knot) that:

$$\begin{aligned}
g(T_{2,-3}(p/q)) &= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \\ \oplus & (t^{-1} & \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \\ \oplus & (t^{1} & \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \end{array} \neq 0 \right\}\\
&= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0 \\ \vee & (t^{-1} & \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \neq 0 \\ \vee & (t^{1} & \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \neq 0 \end{array} \right\} = \dots
\end{aligned}$$

If the second condition is satisfied for $i \in \mathbb{Z}$, the Alexander grading shift implies that the third condition is satisfied for $i+2$, hence the second condition is obsolete.

$$\begin{aligned} \ldots &= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{ll} & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0 \\ \vee\ (t^1 & \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \neq 0 \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0\}, \\ \max\{A \in \mathbb{Z} \mid (t^1\, \widehat{\mathrm{HFK}}(Q_0(p/q)))_{*,A} \neq 0\} \end{array} \right\} = \ldots \end{aligned}$$

We use theorem 4.6 again minding that $Q_{1/2}(p/q)$ is a knot and $Q_0(p/q)$ a two-component link. Observe the additional increment in the second term caused by the grading shift.

$$\ldots = \max\left\{ \begin{array}{l} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + 2 \end{array} \right\}$$

Case II: p is odd and q is even. This works in a similar fashion.
In this case proposition 6.9 gives us

$$\widehat{\mathrm{HFK}}_*\left(T_{2,-3}\left(\frac{p}{q}\right)\right) = \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q)) \\ \oplus & t^{-1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2} \\ \oplus & t^{1} & \widehat{\mathrm{HFK}}_*(Q_0(p/q)) \otimes V^{\otimes 2}. \end{array}$$

We use theorem 4.6 to deduce (mind that $T(p/q)$ is a link) that:

$$\begin{aligned} g(T_{2,-3}(p/q)) &= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \\ \oplus & (t^{-1} & \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \neq 0 \\ \oplus & (t^{1} & \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \end{array} \right\} - 1 \\ &= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{lll} & & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0 \\ \vee & (t^{-1} & \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \neq 0 \\ \vee & (t^{1} & \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \neq 0 \end{array} \right\} - 1 = \ldots \end{aligned}$$

If the second condition is satisfied for $i \in \mathbb{Z}$, the Alexander grading shift implies that the third condition is satisfied for $i+2$, hence the second condition is obsolete.

$$\begin{aligned} \ldots &= \max\left\{ A \in \mathbb{Z} \;\middle|\; \begin{array}{ll} & \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0 \\ \vee\ (t^1 & \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \neq 0 \end{array} \right\} - 1 \\ &= \max\left\{ \begin{array}{l} \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0\}, \\ \max\{A \in \mathbb{Z} \mid (t^1\, \widehat{\mathrm{HFK}}(Q_0(p/q)) \otimes V^{\otimes 2})_{*,A} \neq 0\} \end{array} \right\} - 1 = \ldots \end{aligned}$$

We use theorem 4.6 again minding that $Q_{1/2}(p/q)$ is a two-component link and $Q_0(p/q)$ a knot. Observe the two increments in the second term caused by the grading shift and the tensor product.

$$\ldots = \max \left\{ \begin{array}{l} g(Q_{1/2}(p/q)) + 1, \\ g(Q_0(p/q)) + 2 \end{array} \right\} - 1$$
$$= \max \left\{ \begin{array}{l} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + 1 \end{array} \right\}$$

□

Corollary 6.11. *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with* $p + q \equiv 1 \pmod 2$ *and let*

$$c(p/q) = \begin{cases} 1, & \text{if } p \text{ is odd, } q \text{ is even} \\ 2, & \text{if } p \text{ is even, } q \text{ is odd.} \end{cases}$$

Then we have

$$g\left(T_{2,-3}(p/q)\right) = g\left(Q_0(p/q)\right) + c(p/q).$$

Proof. Observe that lemma 3.22 and 3.23 imply that

$$g\left(Q_0(p/q)\right) + c(p/q) > g(Q_{1/2}(p/q))$$

in both cases.

Case I: p is even and q is odd.

$$g(T_{2,-3}(p/q)) \overset{6.10}{=} \max \left\{ \begin{array}{l} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + 2 \end{array} \right\} \overset{3.22}{=} g(Q_0(p/q)) + 2.$$

Case II: p is odd and q is even.

$$g(T_{2,-3}(p/q)) \overset{6.10}{=} \max \left\{ \begin{array}{l} g(Q_{1/2}(p/q)), \\ g(Q_0(p/q)) + 1 \end{array} \right\} \overset{3.23}{=} g(Q_0(p/q)) + 1.$$

□

Remark 6.12. The last corollary 6.11 and remark 6.6 show that

$$g(T_{2,-3}(p/q)) = 0 \quad \Longrightarrow \quad p/q = 0$$

for all $p/q \in \mathbb{Q}\mathrm{P}^1$ with $p + q \equiv 1 \pmod 2$. Furthermore, as $Q_0(p/q)$ and $Q_0(-p/q)$ are mirror images (and thus have the same genus) corollary 6.11 implies

$$g(T_{2,-3}(p/q)) = g(T_{2,-3}(-p/q))$$

for all $p/q \in \mathbb{Q}\mathrm{P}^1$ with $p + q \equiv 1 \pmod 2$. Using this corollary we can use our tools from section 5 to compute and plot the genus of $g(T_{2,-3}(p/q))$. Maybe useful, but surely nice to look at. See appendices A and B and in particular figure 42. When looking at the figure, note the two statements made in this remark.

Corollary 6.13. *Let $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ with $p+q \equiv 1 \pmod 2$ and even continued fraction $[a_1, \ldots, a_n]$. Then we have*

$$g\left(T_{2,-3}(p/q)\right) = \begin{Bmatrix} 1/2\, n + 1, & \text{if } n \text{ is even} \\ 1/2\,(n+3), & \text{if } n \text{ is odd} \end{Bmatrix} - \begin{Bmatrix} 0, & \text{if } a_1 \neq 0 \text{ or } n \leq 0 \\ 1, & \text{otherwise} \end{Bmatrix}.$$

Proof. Follows from corollary 6.11 and proposition 3.17 minding remark 3.14. □

Corollary 6.14. *Let $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ with $p+q \equiv 1 \pmod 2$. Then*

$$T_{2,-3}(p/q) \text{ is fibred} \iff Q_0(p/q) \text{ is fibred.}$$

Proof. Both rational closures are non-split by remark 6.3 and corollary 2.47, thus by theorem 4.11 it is sufficient to show that

$$\widehat{\mathrm{HFK}}(T_{2,-3}(p/q)) \text{ is monic} \iff \widehat{\mathrm{HFK}}(Q_0(p/q)) \text{ is monic.}$$

Let

$$\begin{aligned} A_{\max} &:= \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(T_{2,-3}(p/q), A) \neq 0\} \\ A_{\max}(1/2) &:= \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(Q_{1/2}(p/q), A) \neq 0\} \\ A_{\max}(0) &:= \max\{A \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(Q_0(p/q), A) \neq 0\}. \end{aligned}$$

From corollary 6.11 and theorem 4.6 we get

$$A_{\max} - \mu(T_{2,-3}(p/q)) + 1 = g(Q_0(p/q)) + c(p/q)$$

and noticing that $c(p/q) + \mu(T_{2,-3}(p/q)) = 3$ (Remark 2.33) this simplifies to

$$A_{\max} = g(Q_0(p/q)) + 2.$$

Observe that the lemmas 3.22 and 3.23 imply

$$g\left(Q_0(p/q)\right) + c(p/q) > g(Q_{1/2}(p/q)) \overset{4.6}{=} A_{\max}(1/2) - \mu(Q_{1/2}(p/q)) + 1$$

hence using $c(p/q) + \mu(Q_{1/2}(p/q)) = 3$ (Remark 2.33) we get

$$g\left(Q_0(p/q)\right) + 2 > A_{\max}(1/2).$$

This shows that

$$A_{\max} > A_{\max}(1/2),$$

which we now apply on proposition 6.8 and 6.9 to see that

$$\dim \widehat{\mathrm{HFK}}_*(T_{2,-3}(p/q), A_{\max}) = \dim \widehat{\mathrm{HFK}}_*(Q_0(p/q), A_{\max}(0)).$$

□

Corollary 6.15. *Let* $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1 \setminus \{0\}$ *with* $p + q \equiv 1 \pmod 2$ *and even continued fraction* $[a_1, \ldots, a_n]$. *Then we have*

$$T_{2,-3}(p/q) \text{ is fibred} \iff \begin{cases} \forall_{i\in\{1,\ldots,n\}} : \ |a_i| = 2, & \text{if } a_n \neq 0, \\ \forall_{i\in\{3,\ldots,n\}} : \ |a_i| = 2, & \text{otherwise.} \end{cases}$$

Proof. Follows from the corollaries 6.14 and 5.21. □

Outlook 6.16. In the past section we specifically examined the rational closures of the (2, -3)-pretzel tangle $T_{2,-3}$. However, the arguments can be - more or less easily - transferred to arbitrary tangles.

Let us consider an oriented Tangle T. We know from [Zib20, Theorem 0.8] that $\mathrm{HFT}(T)$ can be computed combinatorially. We get from theorem 4.24 that $\mathrm{HFT}(T)$ only consists of rational and special components. If we want to compute the knot Floer homology of some rational closure $T(p/q)$, we can use the gluing theorem 4.50 as above. We therefore have to compute a direct sum of Lagrangian intersection Floer homologies between a rational curve of slope p/q and the components of $\mathrm{HFT}(T)$. So far everything is fine.

At this point two issues arise:

1. Can we reduce the pairing with *any* special curves to the paring with (a union of) rational curves plus some bigrading shift or stabilization?
2. Can we control the absolute bigradings of our rational curves in such a way that the relative isomorphisms are indeed absolute when we apply the gluing theorem again for each individual rational pairing in the direct sum?

If the answer to both question is *positive*, we are able to write the knot Floer homology of $T(p/q)$ as direct sum of knot Floer homologies of rational links with possibly shifted bigradings and stabilizations (tensor products with V). In fact, not any rational links, but the p/q-closure of rational tangles. At this point we can formulate a result similar to our main theorem 6.10, which expresses the genus of $T(p/q)$ as the maximum of a set of genera of p/q-closed rational tangles plus some constants which depend on p/q.

Maybe, the genera of these p/q-closures together with the constants can be compared using their even continued fractions, which most probably reduces the number of genera we have to consider for a maximum, but might possibly even give a single determined maximum as in the case of $T_{2,-3}$. Equal arguments can be of course made with other link properties given by the knot Floer homology

So how do we tackle the issues?

Both questions have already been partly answered in this work. By transformation with an element of $\mathrm{Mod}(S^2_4)$ the first question can be reduced to pairings with special curves of slope zero (in this step we need the conjecture 4.47). For such a special curve $\mathfrak{s}_n(0; x, y)$ there are two cases:

1. x and y are pointing in different directions.

2. x and y are pointing in the same direction.

If x and y are pointing in different directions, we can make a similar argument as in lemma 6.4 for arbitrary $n \in \mathbb{N}_{>0}$. Higher n only increase the number of times we have to take the tensor product with V. If x and y are pointing in the same direction, things become more complicated as we cannot compare it with a $\mathfrak{r}(0)$-pairing. A possible answer could be, to look at the pairing with a figure-8 loop (depicted in [Zib20, Figure 48]).

A half answer to the second questions was given in this work by means of the symmetric Alexander grading (4.68, 4.70 and 4.57). If we had similar results for the δ-grading, we would get (absolute) isomorphisms between the Lagrangian Floer homology of (bigrading) symmetrized rational curves and the knot Floer homology of the associated rational link.

A Implementations in Python

```python
#!/usr/bin/env python3

# continuedFractions.py
# by Benedikt Aubeck

# This file contains algorithms concerning the continued fraction of rational tangles
# In particular the proven formulas for cyclical even continued fractions genusOfCECF and leadOfCECF


from utils import divgcd
from math import prod

# Computes the "normal" continued fraction of a rational number t (= [p,q])
def contfrac(t):
    p,q = divgcd(t)
    C = []

    r,rn = p,q
    while rn:
        a = r//rn
        b = r%rn
        C.append(a)
        r,rn = rn,b

    return C

# Computes the even continued continued fraction of rational number t (= [p,q]) with p+q = 1 (mod 2)
def contfrac2(t):
    p,q = divgcd(t)
    assert p%2 + q%2 == 1, "Wrong parities!"
    C = []

    r,rn = p,q
    while rn:
        a = r//rn
        if a%2 == 1:
            a = a+1
        b = r-a*rn
        C.append(a)
        r,rn = rn,b

    return C

# Evaluates a continued fraction C (= [a_1,a_2,...,a_n]) to its rational number [p,q]
def simplcf(C):
    num, den = 1, 0
    for u in reversed(C):
```

```python
        num, den = den + num*u, num

    num,den = divgcd([num,den])
    return num, den


# Computes the genus of a rational link with cyclical even continued fraction C (= [a_1,a_2,...,a_n])
def genusOfCECF(C):
    n = len(C)
    if n%2 == 0:
        r = n/2
    else:
        r = (n-1)/2

    if C[0] == 0 and n > 0:
        r = r-1

    return int(r)

# Given a rational number t (= [p,q]), this computes the genus of the numerator of the rational tangle with slope p/q. If this numerator is a link it computes the genera of both orientations.
def genusOfNumerator(t):
    p,q = divgcd(t)

    if (p+q)%2 == 0:
        q += p

    if p%2 == 1:
        C = contfrac2([p,q])
        r = genusOfCECF(C)

        return int(r), None

    elif p%2 == 0:
        C1 = contfrac2([p,q])
        C2 = contfrac2([p,q+p])
        r1 = genusOfCECF(C1)
        r2 = genusOfCECF(C2)

        return int(r1), int(r2)

# Computes the number of leading (hightest Alexander degree) generators of the knot Floer homolgy of a rational link with cyclical even continued fraction C (= [a_1,a_2,...,a_n])
def leadOfCECF(C):
    n = len(C)
    if n == 1 and C[0] == 0:
        l = 2
    elif n > 1 and C[0] == 0:
        tmp = [abs(x)/2 for x in C[2:]]
        if len(tmp) == 0:
```

```
                l = 0
            else:
                l = prod(tmp)
    else:
        tmp = [abs(x)/2 for x in C]
        l = prod(tmp)

    return int(l)

# Given a rational number t (= [p,q]), this computes the number of leading (
    hightest Alexander degree) generators of the knot Floer homolgy of the
    numerator of the rational tangle with slope p/q. If this numerator is a
    link it computes it for both orientations.
def leadOfNumerator(t):
    p,q = divgcd(t)

    if (p+q)%2 == 0:
        q += p

    if p%2 == 1:
        C = contfrac2([p,q])
        r = leadOfCECF(C)

        return int(r), None

    elif p%2 == 0:
        C1 = contfrac2([p,q])
        C2 = contfrac2([p,q+p])
        r1 = leadOfCECF(C1)
        r2 = leadOfCECF(C2)

        return int(r1), int(r2)
```

code/continuedFractions.py

```
#!/usr/bin/env python3

# HFK.py
# by Benedikt Aubeck

# This file contains a class to represent the knot Floer homolgy of oriented
    links
# Initialisation:
#   ngen    the dimension
#   dG      a list of delta degrees
#   dA      a list of Alexander degrees (gets matched in order to dG)
#   isknot  a boolean saying whether it is the HFK of a knot

class HFK:
    def __init__(self, ngen, dG, aG, isknot=None):
        self.ngen = ngen
        self.bgvs = [[None,None] for i in range(self.ngen)]

        self.dG = dG
```

```
        self.aG = aG

        self.isknot = isknot

        self.set_dG(self.dG)
        self.set_aG(self.aG)

        assert len(self.dG) == self.ngen and len(self.aG) == self.ngen

    # Set the delta grading
    def set_dG(self, dG):
        self.dG = dG
        for i in range(self.ngen):
            self.bgvs[i][0] = self.dG[i]

    # Set the Alexander grading
    def set_aG(self, aG):
        self.aG = aG
        self.aG.sort()
        for i in range(self.ngen):
            self.bgvs[i][1] = self.aG[i]

    # Shift the delta grading by n and the Alexander grading by m
    def shift(self, n, m):
        for i in range(self.ngen):
            self.dG[i] += n
            self.dA[i] += m

    def __repr__(self):
        return str(self.bgvs)

    # Simulates the tensor product with V
    # i.e. each generator gets doubled in the same delta grading
    # and with Alexander gradings shifted by -1/2 and +1/2
    def V(self, k=1, shifted=False):
        ngen = 2*k*self.ngen

        dG = []
        for d in self.dG:
            for _ in range(k):
                dG.append(d)
                dG.append(d)

        aG = []
        for a in self.aG:
            for i in range(k):
                if not shifted:
                    aG.append(a-1/2)
                    aG.append(a+1/2)
                else:
                    aG.append(a-1/2+i)
                    aG.append(a+1/2+i)
```

```
        return HFK(ngen, dG, aG)

    # Maximal Alexander degree - mimial Alexander degree
    def width(self):
        return max(self.aG) - min(self.aG)

    # This method symmetrized the Alexander gradings (for real HFK's this is
    possible)
    def symm(self):
        offset = min(self.aG) + self.width()//2
        self.set_aG([int(x-offset) for x in self.aG])

    # Returns the genus of the theoretical oriented link
    def genus(self):
        m = max(self.aG)
        if not float(m).is_integer():
            print(f"{max(self.aG)} is no integer!")
        if self.isknot:
            return int(m)
        else:
            if self.ngen == 2:
                return 0

            return int(m)-1

    # Returns number of generators in the highest Alexander degree
    def numleadinggenerators(self):
        m = max(self.aG)
        return sum([1 if self.aG[i] == m else 0 for i in range(self.ngen)])

    # Returns wheter the theoretical oriented link is fibred
    def isfibred(self):
        if self.numleadinggenerators() == 1:
            return True
        else:
            return False
```

code/HFK.py

```
#!/usr/bin/env python3

# HFKOfTwoBridgeLinks.py
# by Benedikt Aubeck

# This file contains the main method effHFK to compute the HFK of oriented
    rational links (up to a delta grading shift)
# Moreover there are two helper function, in particular the algorithm to
    numertorize a rational link

from utils import divgcd, extgcd
from HFK import HFK

# This is the "Numeratorize"-lemma in the work
def numeratorize(t,s):
```

```
    t = divgcd(t)
    s = divgcd(s)

    _, a, b = extgcd(t[0],t[1])
    a,b = -a,-b

    q = -b*s[1] + a*s[0]
    p = t[0]*s[1] + t[1]*s[0]

    return divgcd([p,q])

# This is a little bit complicated
# It takes the slope t (=[p,q]) of a rational tangle and returns two function y and yinv
# y     takes as argument i = 1,2,3 or 4 and returns the tangle end that i gets map to under the "numeratorize"-transformation
# yinv  is the inverse to y
def morphhelper(t):
    t = divgcd(t)

    _, a, b = extgcd(t[0],t[1])
    a,b = -a,-b

    def f(x):
        if x == [0,0]:
            return 1
        elif x == [0,1]:
            return 2
        elif x == [1,1]:
            return 3
        elif x == [1,0]:
            return 4

    l = list(map(f,[[0,0], [a%2, t[1]%2], [(a-b)%2, (t[0]+t[1])%2], [-b%2, t[0]%2]]))

    def y(i):
        return l[i-1]

    def yinv(i):
        for j in range(4):
            if l[j] == i:
                return j+1

    return y,yinv



# The methods computs the HFK (up to a delta grading shift) of the oriented rational link given by the
# union of the two rational tangles given by the slope T (=[t1, t2]) and Q (=[q1,q2]) and the orientation s
# The orentation works as follows: Think about the oriented union of (the
```

```
    rational tangles of slope) T and S.
#   One of the two tangles is oriented such that the distinguished tangle
    end 1 is pointing inwards.
#   We call this tangle Q.
#   This fixes the orientation of one component.
#   The orientation s (=[x,y]) then must specify the orientation of the
    other component.
#   We do this by giving the two tangles ends x,y of Q not connected to 1 as
    list [x,y] where x is the inwards pointing end.

def effHFK(T, Q, s):
    flag = 0
    y, _ = morphhelper(T)
    if list(map(y,s)) == [2,3]:
        flag = 1

    firstone = True

    ### 1
    p, q = numeratorize(T, Q)
    if not (p == 0 or ((-q)//p + p+q + flag)%2 == 0):
        firstone = False


    if q == 0:
        return HFK(1, [0], [0], True)
    elif p == 0:
        return HFK(2, [0,1], [0,0], False)

    ### 2
    if p < 0:
        p,q = -p,-q

    q = q%p
    if p%2+q%2 == 2:
        q = p-q

    ### 3
    if p%2 == 1 or firstone:
        lamb = [(y*q)//p for y in range(2*p)]
        Gamma = [-lamb[y]%2 * (-1)**y for y in range(2*p)]
        mem = [0]
        for y in range(1, 2*p):
            mem.append(mem[y-1] + Gamma[y] + Gamma[y-1])
        Delta = [mem[k] for k in range(0, 2*p, 2)]

    else:
        lamb = [(y*q)//p for y in range(2*p)]
        Gamma = [lamb[y]%2 for y in range(2*p)]
        mem = [0]
        for y in range(1, 2*p):
            mem.append(mem[y-1] + Gamma[y] + Gamma[y-1])
        Delta = [-k + mem[k] for k in range(0, 2*p, 2)]
```

```
    ### 4 & 5
    A = []
    for k in range(p):
        A.append(Delta[k])

    if p%2 ==1:
        hfk = HFK(p, [0]*p, A)
        hfk.isknot = True
    else:
        hfk = HFK(p, [0]*p, A).V()
        hfk.isknot = False

    hfk.symm()
    return hfk
```

code/HFKOfTwoBridgeLinks.py

```
#!/usr/bin/env python3

# comparegenus.py
# by Benedikt Aubeck

# This file contains a method which simplifies the process of comparing many rational closures of two rational tangles according to their genera

import numpy as np
import matplotlib.pyplot as plt
from random import randint

from utils import divgcd
from HFKOfTwoBridgeLinks import effHFK
from graphicalRepresentation import setup, drawline

# Compares rational p/q-closures of the rational tangles T1 (=[p1,q1]) and T2 (=[p2,q2]) with given orientations o1 and o2 and plots a diagram

# On the orientations:
#   For the given tangle T we assume that the distinguished tangle end 1 is pointing inwards.
#   This fixes the orientation of the leading string.
#   The orientation o (=[x,y]) then must specify the orientation of the other string.
#   We do this by giving the other two tangles ends x,y as list [x,y] where x is the inwards pointing end.
#   E.g. The leading string connects 1-4. Then the other string connects 2-3.
#         If we want the orientation where 3 is pointing inwards, the orientation would be [3,2].

# pars   which parities are allows for p and q
#        [1,0] p odd, q even
#        [0,1] p even, q odd
#        [1,1] p odd, q odd
```

```
#           None    all p and q

# n         the number of closures to compute

# res       p and q are picked from the interval [-res, res]
def comparegenus(T1, o1, T2, o2, pars = None, n = 1000, res = 1000):
    fig, ax = plt.subplots()
    fig.suptitle(f"{T1[0]}/{T1[1]} (o. {o1}) vs. {T2[0]}/{T2[1]} (o. {o2})\n
    pars = {pars}, n = {n}, res = {res}\n GENUS")

    listoflines0 = []
    size = 0

    for i in range(n):
        print(i)

        flag = False
        while not flag:
            p = randint(-res, res)
            q = randint(-res, res)

            if p == 0 and q == 0:
                continue

            p,q = divgcd([p,q])

            if (pars is None):
                flag = True
            elif (p%2 == pars[0] and q%2 == pars[1]):
                flag = True

        l1 = effHFK([-p,q], T1, o1)
        l2 = effHFK([-p,q], T2, o2)

        diff = l1.genus()-l2.genus()

        # Normalize to show only the sign of the difference
        # if diff > 0:
        #     diff = 1
        # elif diff < 0:
        #     diff = -1

        line = [[p,q], l1.genus()-l2.genus()]
        listoflines0.append(line)
        size = max(abs(line[1]), size)

    setup(fig, ax, size, square=True, circles=False)
    hsv = plt.get_cmap('hsv')
    cm = hsv(np.linspace(0, 1, size+1))

    for line in listoflines0:
        drawline(ax, line[0], line[1], cm, showzero=True)
```

```
    return ax
```

code/comparegenus.py

```
#!/usr/bin/env python3

# comparelead.py
# by Benedikt Aubeck

# This file contains a method which simplifies the process of comparing many rational closures of two rational tangles according to their number of leading generators in their knot Floer homolgies

import numpy as np
import matplotlib.pyplot as plt
from random import randint

from utils import divgcd
from HFKOfTwoBridgeLinks import effHFK
from graphicalRepresentation import setup, drawline

# Compares rational p/q-closures of the rational tangles T1 (=[p1,q1]) and T2 (=[p2,q2]) with given orientations o1 and o2 and plots a diagram

# On the orientations:
#    For the given tangle T we assume that the distinguished tangle end 1 is pointing inwards.
#    This fixes the orientation of the leading string.
#    The orientation o (=[x,y]) then must specify the orientation of the other string.
#    We do this by giving the other two tangles ends x,y as list [x,y] where x is the inwards pointing end.
#    E.g. The leading string connects 1-4. Then the other string connects 2-3.
#          If we want the orientation where 3 is pointing inwards, the orientation would be [3,2].

# pars    which parities are allows for p and q
#         [1,0] p odd, q even
#         [0,1] p even, q odd
#         [1,1] p odd, q odd
#         None  all p and q

# n       the number of closures to compute

# res     p and q are picked from the interval [-res, res]
def comparelead(T1, o1, T2, o2, pars = None, n = 5000, res = 1000):
    fig, ax = plt.subplots()
    fig.suptitle(f"{T1[0]}/{T1[1]} (o. {o1}) vs. {T2[0]}/{T2[1]} (o. {o2})\n pars = {pars}, n = {n}, res = {res}\n LEADING GENERATORS")

    listoflines0 = []
    size = 0
```

```
    lofrat = []

    for i in range(n):
        print(i)

        flag = False
        while not flag:
            p = randint(-res, res)
            q = randint(-res, res)

            if p == 0 and q == 0:
                continue

            p,q = divgcd([p,q])

            if (pars is None):
                flag = True
            elif (p%2 == pars[0] and q%2 == pars[1]):
                flag = True

        l1 = effHFK([-p,q], T1, o1)
        l2 = effHFK([-p,q], T2, o2)

        diff = l1.numleadinggenerators()-l2.numleadinggenerators()

        # Normalize to show only the sign of the difference
        if diff > 0:
            diff = 1
        elif diff < 0:
            diff = -1

        line = [[p,q], diff]
        listoflines0.append(line)
        size = max(abs(line[1]), size)

    setup(fig, ax, size, square=True, circles=False)
    hsv = plt.get_cmap('hsv')
    cm = hsv(np.linspace(0, 1, size+1))

    for line in listoflines0:
        drawline(ax, line[0], line[1], cm, showzero=True)

    return ax
```

code/comparelead.py

```
#!/usr/bin/env python3

# utils.py
# by Benedikt Aubeck

# This file contains some helper methods

from math import gcd
```

```python
# Extended greatest common divisor algorithm
# Returns: a = gcd(a,b)
#          u, v such that u*a + v*b = gcd(a,b)
def extgcd(a, b):
    u, v, s, t = 1, 0, 0, 1
    while b!=0:
        q = a//b
        a, b = b, a-q*b
        u, s = s, u-q*s
        v, t = t, v-q*t
    return a, u, v

# A given pair [p,q] is reduced to resemble a rational number
# Futhermore, it shifts a potential minus sign into the first entry
def divgcd(l):
    if l[0] == 0:
        return [0,1]
    if l[1] == 0:
        return [1,0]

    l = [l[0]//gcd(l[0],l[1]), l[1]//gcd(l[0],l[1])]
    if l[1] < 0:
        l[0],l[1] = -l[0],-l[1]

    return l

# Read with prompt s the fraction of a rational tangle
# Possible inputs are e.g. "1" "1/2" "1 2"
def gettangle(s):
    t = input(s)
    l = t.split()
    if len(l) == 1:
        l = l[0].split('/')
    if len(l) == 1:
        l.append('1')
    for i in range(2):
        try:
            l[i] = int(l[i])
        except:
            print("Oops!  That were no valid numbers.")
            exit(1)

    l = divgcd(l)
    return l
```

code/utils.py

```python
#!/usr/bin/env python3

# graphicalrepresentation.py
# by Benedikt Aubeck
```

```python
# This file contains the methods used to plot the wanted (and maybe useful) diagrams

import matplotlib.pyplot as plt
from matplotlib.lines import Line2D
import numpy as np

# Calculate the cartesian coordinates out of polar coordinates (rho, phi).
# The flag showzero is set to True if we want to visualize vectors with lenght zero (sounds funny but works)
def pol2cart(rho, phi, showzero):
    if showzero and rho == 0:
        rho = 1/2
    x = rho * np.cos(phi)
    y = rho * np.sin(phi)
    return(x, y)


# Sets up the plot with figure fig, axes ax. The size is for both coordinates
# half=True      focus on the upper half of the diagram (y>0
# square=True    centers the plot on the origin (instead of only the half plane x>0)
# circles=False  deactivate the circles showing the radial distance from the origin
def setup(fig, ax, size, half=False, square=False, circles=True):
    if square:
        fig.set_size_inches(10, 10)
    else:
        fig.set_size_inches(5, 10)

    if circles:
        circle = []
        for i in range(size):
            circle.append(plt.Circle((0, 0), radius=i+1, edgecolor='black', fill=False))
            ax.add_patch(circle[i])

    ax.set_aspect('equal')
    #ax.grid(True, 'major')
    if square:
        ax.set_xlim(-size, size)
    else:
        ax.set_xlim(0, size)

    ax.set_xticks([])

    if not half:
        ax.set_ylim(-size, size)
        ax.set_yticks(np.arange(-size,size+1))
        ax.set_yticklabels(np.abs(np.arange(-size,size)))
    else:
        ax.set_ylim(0, size)
        ax.set_yticks(np.arange(0,size+1))
```

```
        ax.set_yticklabels(np.abs(np.arange(0,size)))


# Draw a line with slope t (=[p,q]) and integer lenght r onto axes ax
# cm              is the used colormap
# showzero=True   indicate lines with lenght zero by drawing them with length 1/2
def drawline(ax, t, r, cm, showzero=False):
    if t[1] != 0:
        m = np.arctan(t[0]/t[1])
        x, y = pol2cart(r, m, showzero)
    else:
        x, y = 0, r

    if showzero and r == 0:
        line = Line2D([0, x], [0, y], color='black', alpha=0.5, linewidth=0.5)
    else:
        line = Line2D([0, x], [0, y], color=cm[-abs(r)], alpha=0.5, linewidth=0.5)

    ax.add_line(line)
```

code/graphicalRepresentation.py

B Visualizations

The following example program uses the implementations from above to compare rational closures of the two oriented rational tangles Q_0 and $Q_{1/2}$ depicted in figure 39. This graphical hint (figures 40 and 41) has led to the lemmas 3.22 and 3.23. A short explanation of the plots: For a given $p/q \in \mathbb{Q}\mathrm{P}^1$ we compute

$$r := g(Q_0(p/q)) - g(Q_{1/2}(p/q))$$

and draw a coloured line segment from the origin with slope p/q and length $|r|$. If r is positive it is drawn into the right half-plane and in left half-plane if r is negative. For $p/q = \infty$ "up" is the positive direction. In case r is zero we draw a black line segment with length $1/2$ in positive direction.

```
#!/usr/bin/env python3

# compareZeroAndOnehalf.py
# by Benedikt Aubeck

# This files contains an example on how to use the implementations

import matplotlib.pyplot as plt

from comparegenus import comparegenus
from comparelead import comparelead

```

```
ax = comparegenus([0,1], [3,2], [1,2], [3,4], pars=[1,0], n=3000, res=1000)
#ax = comparelead([0,1], [3,2], [1,2], [3,4], pars=[1,0], n=3000, res=1000)

ax = comparegenus([0,1], [3,2], [1,2], [3,4], pars=[0,1], n=3000, res=1000)
#ax = comparelead([0,1], [3,2], [1,2], [3,4], pars=[0,1], n=3000, res=1000)

plt.show()
```

code/compareZeroAndOnehalf.py

The next program visualizes (Figure 42) the concluding result of section 6.3, in particular corollary 6.11. The program plots the genus $g(T(p/q))$ for $p/q \in \mathbb{Q}P^1$ with $p+q \equiv 1 \pmod 2$ as a line segment with slope p/q starting from the origin with length $g(T(p/q))$. The parameter "res" specifies which random $p/q \in \mathbb{Q}P^1$ to pick, namely such that $p, q \in [-\mathrm{res}, \mathrm{res}]$. The parameter "n" sets the number of random fraction to plot.

```
#!/usr/bin/env python3

# TwoMinusThreePretzelTangle.py
# by Benedikt Aubeck

# This file plots the genus of rational closures of the (2,-3)-pretzel tangle

import numpy as np
import matplotlib.pyplot as plt
from random import randint

from utils import divgcd
from HFKOfTwoBridgeLinks import effHFK
from graphicalRepresentation import setup, drawline

if __name__ == "__main__":
    fig, ax = plt.subplots()
    res = 5
    n = 100

    listoflines0 = []
    size = 0

    for i in range(n):
        print(i)

        flag = False
        while not flag:
            p = randint(-res, res)
            q = randint(-res, res)

            if p == 0 and q == 0:
                continue

            p,q = divgcd([p,q])

            if (p+q)%2 == 1:
```

```
                flag = True

        l1 = effHFK([0,1], [-p,q], [4,1])

        if p == 0:
            g = 0
        elif p%2 == 1 and q%2 == 0:
            g = l1.genus() + 1
        elif p%2 == 0 and q%2 == 1:
            g = l1.genus() + 2

        line = [[p,q], g]
        listoflines0.append(line)
        size = max(abs(line[1]), size)

    setup(fig, ax, size, square=False, circles=False)
    ax.set_yticks([])
    hsv = plt.get_cmap('hsv')
    cm = hsv(np.linspace(0, 1, size+1))

    for line in listoflines0:
        drawline(ax, line[0], line[1], cm, showzero=True)

    plt.show()
```

code/TwoMinusThreePretzelTangle.py

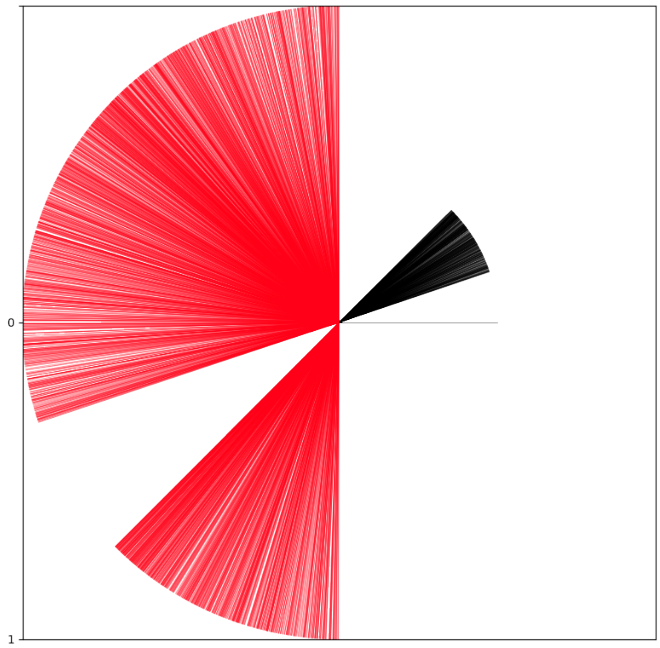


Figure 40: $g(Q_0(p/q)) - g(Q_{1/2}(p/q))$ for p even, q odd

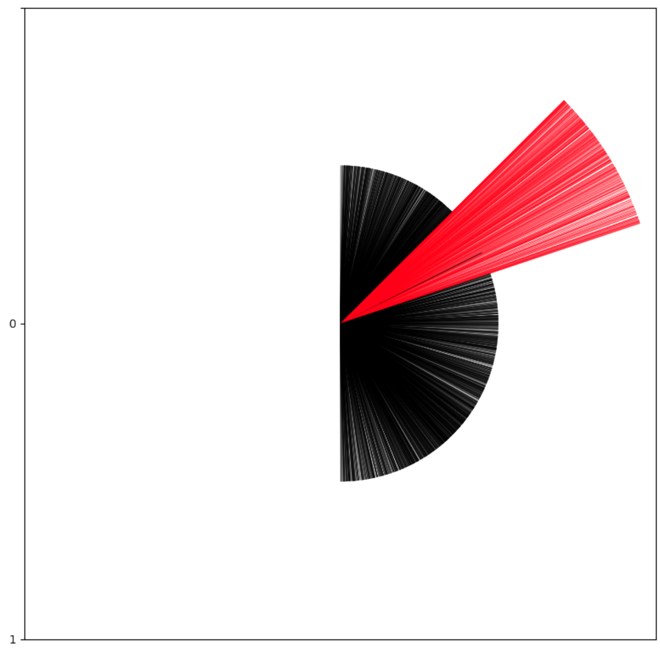
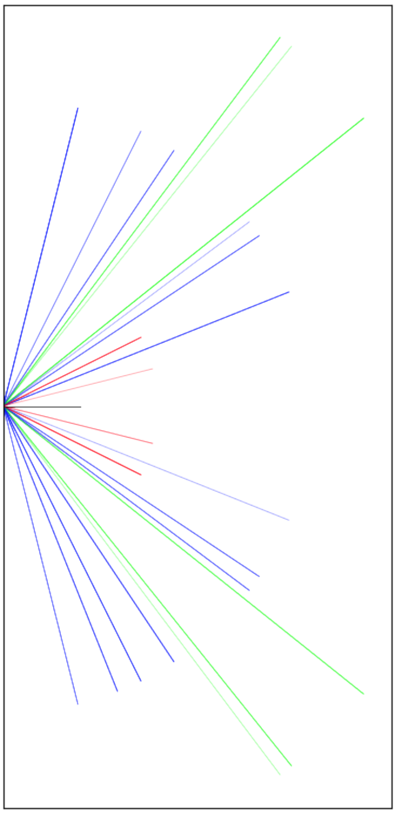
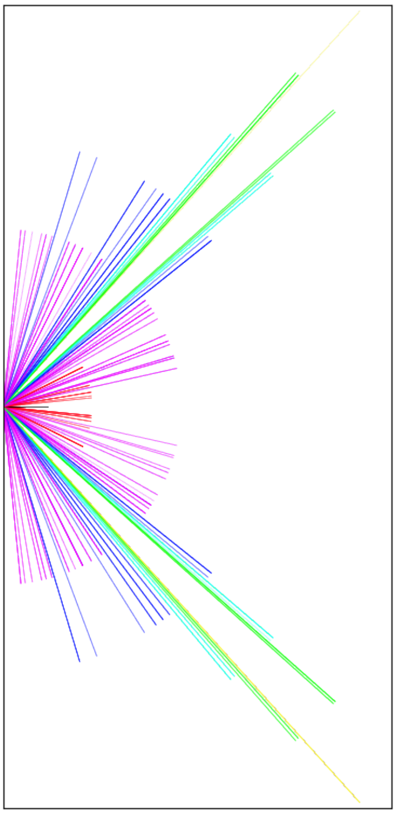


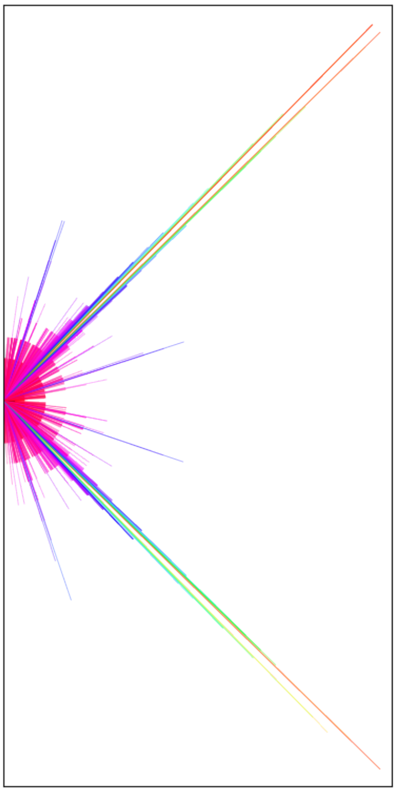
Figure 41: $g(Q_0(p/q)) - g(Q_{1/2}(p/q))$ for p odd, q even



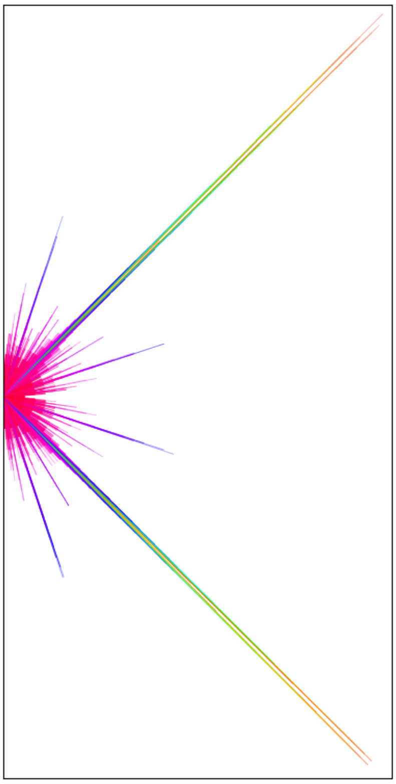
(a) res = 5, n = 100



(b) res = 10, n = 500



(c) res = 50, n = 1000



(d) res = 100, n = 5000

Figure 42: Plotted $g(T_{2,-3}(^{p}/_{q}))$ for different parameters

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