Thin links and Conway spheres

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joint work in progress with Artem Kotelskiy and Liam Watson

University of British Columbia

A notion of minimality for homology theories

Singular homology

Heegaard Floer homology

Characterizing L-spaces

Minimality for link homology theories

Knot Floer homology

Khovanov homology

A-links vs. thin links

Characterization of A-link fillings

A-link gluing theorem

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Invariants for four-ended tangles

The δ-grading on the tangle invariants

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A notion of minimality for homology theories

H_* some homology theory, e.g.

- singular homology,
- Heegaard Floer homology,
- Khovanov homology,
- etc.

Observation

$$\dim \mathsf{H}_*(Y) \geq \left| \textit{Euler char. } \chi\, \mathsf{H}_*(Y) = \sum_i (-1)^i \dim H_i(Y) \right|$$

Problem

Characterize those objects Y for which $\dim \mathsf{H}_*(Y) = |\chi\, \mathsf{H}_*(Y)|$.

Equivalently: Classify objects Y for which $\mathsf{H}_*(Y)$ is supported in degrees of the same parity.

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Minimality for singular homology

$\mathsf{H}_* = \mathsf{H}_*(-; \mathbf{k})$ = singular homology

Problem

Characterize all n-manifolds Y for which $\dim \mathsf{H}_*(Y) = |\chi\, \mathsf{H}_*(Y)|$.

- n odd: $\chi\, \mathsf{H}_*(Y) = 0$, so there is no such Y.
- $n = 2$:
  - if orientable, then $Y = S^2$;
  - if non-orientable and $\mathrm{char}(\mathbf{k}) \neq 2$, then $Y = \mathbb{R}P^2$
- even $n > 2$:
  - naïve guess: Y should have only i-handles for even i
  - open question [Kirby problem 4.18]: Does every closed, simply-connected 4-manifold admit a handle-decomposition without 1-handles and 3-handles?
  - for $n > 4$: ???

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Minimality for Heegaard Floer homology

$\widehat{\mathrm{HF}}$ = Ozsváth-Szabó's Heegaard Floer homology of 3-manifolds

Problem

Characterize all 3-manifolds Y for which $\dim \widehat{\mathrm{HF}}(Y) = |\chi\, \widehat{\mathrm{HF}}(Y)|$.

Solutions are called L-spaces.

- Examples: **L**ens spaces (hence the name)
- $\chi\, \widehat{\mathrm{HF}}(Y) = \begin{cases} |H_1(Y;\mathbb{Z})| & \text{if } b_1(Y) = 0 \\ 0 & \text{if } b_1(Y) > 0 \end{cases}$
- All L-spaces are rational homology spheres (i.e. $b_1(Y) = 0$).
- L-space conjecture: [Boyer-Gordon-Watson]

Heegaard Floer theory (non-L-space) ↔ codimension 1 structures (taut foliations)

Heegaard Floer theory (non-L-space) ↔ fundamental group (left-orderability)

codimension 1 structures (taut foliations) ↔ fundamental group (left-orderability)

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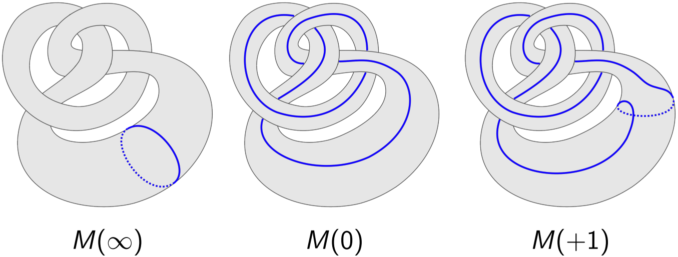
Characterizing L-spaces

Question

When does Dehn surgery along a knot give an L-space?



Theorem [Rasmussen-Rasmussen'16]

For any 3-manifold M with torus boundary,

$$\mathcal{L}(M) := \{s \in \mathbb{Q}\mathrm{P}^1 \mid M(s) \text{ is an L-space}\}$$

is either $\varnothing$, a single point, a closed interval or $\mathbb{Q}\mathrm{P}^1 \setminus \{\lambda\}$.

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Theorem [Hanselman-Rasmussen-Watson'16]

Let M and M' be two 3-manifolds with torus boundary and $h\colon \partial M \xrightarrow{\cong} \partial M'$, which are boundary incompressible. Then

$$M \cup_h M' \textit{ is an L-space} \Leftrightarrow h(\mathring{\mathcal{L}}(M)) \cup \mathring{\mathcal{L}}(M') = \mathbb{Q}\mathrm{P}^1 .$$

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Minimality for link homology theories

Problem

Characterize those links L for which $\dim \mathrm{H}_*(L) = |\chi \, \mathrm{H}_*(L)|$.

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Minimality for knot Floer homology

$\widehat{\text{HFK}}$ = knot Floer homology of an ℓ-component link

$$\widehat{\text{HFK}}(L) = \bigoplus_{\substack{h\in\mathbb{Z} \leftarrow \text{homological (Maslov) grading} \\ A\in\mathbb{Z} \leftarrow \text{Alexander grading}}} \widehat{\text{HFK}}(L, A, h)$$

$$\chi_{gr}\,\widehat{\text{HFK}}(L) = \sum_{h,A} (-1)^h t^A \dim \widehat{\text{HFK}}(L, A, h) = \underbrace{\Delta_L(t)}_{\text{Alexander polynomial}} \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\ell-1}$$

Problem

Characterize all links $L \subset S^3$ for which $\dim \widehat{\text{HFK}}(L) = |\chi\,\widehat{\text{HFK}}(L)|$.

$$\chi\,\widehat{\text{HFK}}(L) = \sum_{h,A} (-1)^h \dim \widehat{\text{HFK}}(L, A, h) = \begin{cases} \Delta_L(1) = 1 & \text{if } \ell = 1 \\ 0 & \text{if } \ell > 1 \end{cases}$$

- for $\ell = 1$: one solution (unknot detection [Ozsváth-Szabó'03])
- for $\ell > 1$: no solution

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Minimality for Khovanov homology

$\widetilde{\mathrm{Kh}}$ = reduced Khovanov homology of an ℓ-component link

$$\widetilde{\mathrm{Kh}}(L) = \bigoplus_{\substack{h\in\mathbb{Z} \leftarrow \text{(co)homological grading} \\ q\in\mathbb{Z} \leftarrow \text{quantum grading}}} \widetilde{\mathrm{Kh}}(L,q,h)$$

- $\chi_{gr}\widetilde{\mathrm{Kh}}(L) = \sum_{h,q}(-1)^h t^q \dim \widetilde{\mathrm{Kh}}(L,q,h) = \underbrace{V_L(t)}_{\text{Jones polynomial}}$

Problem

Characterize those links $L \subset S^3$ for which $\dim \widetilde{\mathrm{Kh}}(L) = |\chi \widetilde{\mathrm{Kh}}(L)|$.

- $\chi\widetilde{\mathrm{Kh}}(L) = \sum_{h,q}(-1)^h \dim \widetilde{\mathrm{Kh}}(L,q,h) = V_L(1) = 2^{\ell-1}$
- $\ell = 1$: one solution (unknot detection [Kronheimer-Mrowka'10])
- $\ell > 1$: complete classification (forest of unknots [Xie-Zhang'19])

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Khovanov A-links

Theorem [Ozsváth-Szabó'03]

For any link $L \subset S^3$, $\exists$ *spectral sequence*

$$\widetilde{\mathrm{Kh}}(\mathrm{m}\,L;\mathbb{F}) \rightrightarrows \widehat{\mathrm{HF}}(\Sigma(L);\mathbb{F}).$$

Hence

$$\dim \widetilde{\mathrm{Kh}}(L;\mathbb{F}) \geq \dim \widehat{\mathrm{HF}}(\Sigma(L);\mathbb{F}) \geq |H_1(\Sigma(L))| = \det(L).$$

Definition [Kotelskiy-Watson-Z'20]

We call a link L a **Khovanov A-link** if $\dim \widetilde{\mathrm{Kh}}(L) = \det(L)$.

- L is a Khovanov A-link $\Rightarrow \Sigma(L)$ is an L-space
- Examples: **A**lternating links (hence the name)

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Khovanov A-links

Definition [Kotelskiy-Watson-Z'20]

We call a link L a **Khovanov A-link** if $\dim \widetilde{\mathrm{Kh}}(L) = \det(L)$.

For any link L, $\det(L) \overset{\text{def}}{=} |\Delta_L(-1)| \overset{\text{[Jones'85]}}{=} |V_L(-1)|$.

$$\chi_{gr} \widetilde{\mathrm{Kh}}(L) = \sum_{h,q} (-1)^h t^q \dim \widetilde{\mathrm{Kh}}(L, q, h) = V_L(t)$$

$$\chi \widetilde{\mathrm{Kh}}_\delta(L) = \sum_{\delta = q-h} (-1)^\delta \dim \widetilde{\mathrm{Kh}}(L, q, h) = V_L(-1)$$

So A-links are precisely the solutions to:

Problem

Characterize those links $L \subset S^3$ *for which* $\dim \widetilde{\mathrm{Kh}}(L) = |\chi \widetilde{\mathrm{Kh}}_\delta(L)|$.

Heegaard Floer A-links

Definition [Kotelskiy-Watson-Z'20]

A link L is a **Heegaard Floer A-link** if $\dim \widehat{\mathsf{HFK}}(L) = 2^{\ell-1}\det(L)$.

- Examples: **A**lternating links (hence the name)

$$\chi_{gr}\,\widehat{\mathsf{HFK}}(L) = \sum_{h,A}(-1)^h t^A \dim \widehat{\mathsf{HFK}}(L,A,h) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\ell-1}\cdot\Delta_L(t)$$

$$\chi\,\widehat{\mathsf{HFK}}_\delta(L) = \sum_{\delta=A-h}(-1)^\delta \dim \widehat{\mathsf{HFK}}(L,A,h) = 2^{\ell-1}\underbrace{\Delta_L(-1)}_{=\pm\det(L)}$$

So A-links are precisely the solutions to:

Problem

Characterize all links $L \subset S^3$ *with* $\dim \widehat{\mathsf{HFK}}(L) = |\chi\,\widehat{\mathsf{HFK}}_\delta(L)|$.

Heegaard Floer vs. Khovanov A-links

Theorem [Dowlin'18]

For any knot L in S^3, $\exists$ spectral sequence

$$\widetilde{\mathrm{Kh}}(\mathrm{m}\,L;\mathbb{Q}) \rightrightarrows \widehat{\mathrm{HFK}}(L;\mathbb{Q}).$$

Hence

$$\dim \widetilde{\mathrm{Kh}}(L;\mathbb{Q}) \geq \dim \widehat{\mathrm{HFK}}(L;\mathbb{Q})$$

More generally, for links L,

$$2^{\ell-1} \dim \widetilde{\mathrm{Kh}}(L;\mathbb{Q}) \geq \dim \widehat{\mathrm{HFK}}(L;\mathbb{Q})$$

L is a Khovanov A-link $\implies$ L is a Heegaard Floer A-link

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More generally, for links L,

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L is a Khovanov A-link $\overset{?}{\Longleftrightarrow}$ L is a Heegaard Floer A-link

What are A-links?

Let $\mathrm{H}_* = \widetilde{\mathrm{Kh}}_\delta$ or $\widehat{\mathrm{HFK}}_\delta$.

Definition

A link L is a **thin** if $\mathrm{H}_*(L)$ is supported in a single δ-grading.

So every thin link is an A-link.

Conjecture [folklore]

For any link L, $\mathrm{H}_*(L)$ has full support, i.e. for any $i < j < k$,

$$\Big(\mathrm{H}_i(L) \neq 0 \quad \textit{and} \quad \mathrm{H}_k(L) \neq 0\Big) \Rightarrow \mathrm{H}_j(L) \neq 0.$$

If this is true, then A-links are precisely thin links.

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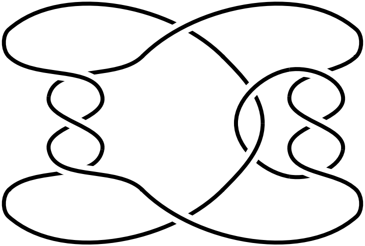

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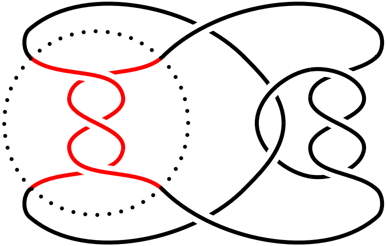

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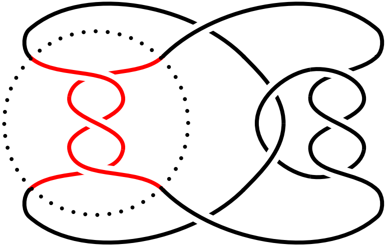
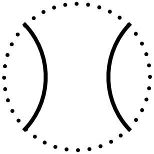


Theorem [John Conway (1937–2020)]

$$\{\textit{rational tangles } Q_s\} \leftrightarrow \mathbb{Q}\mathrm{P}^1$$



Q_∞

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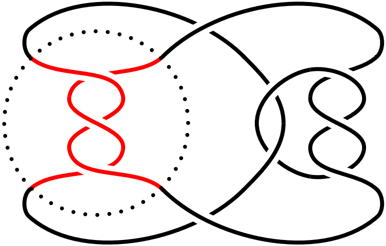

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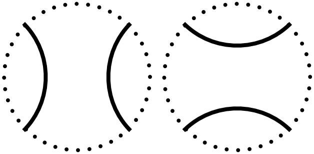

Q_∞ Q_0

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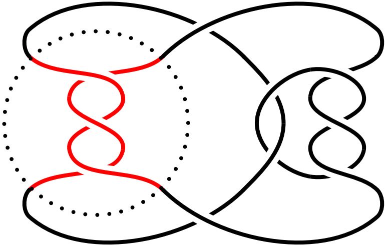

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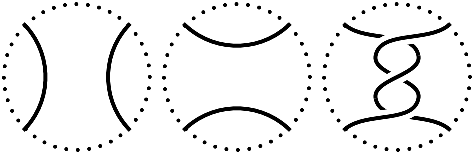

Q_∞ Q_0 $Q_{1/3}$

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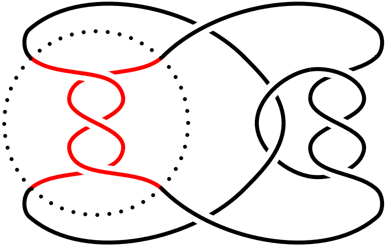

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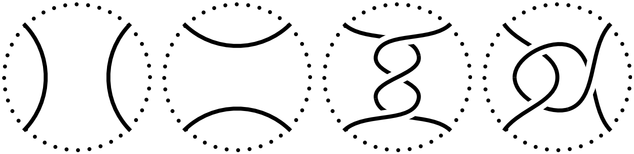

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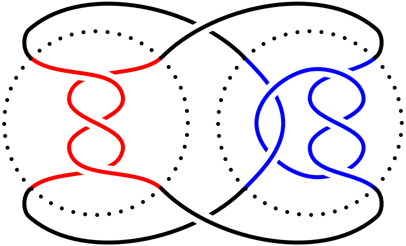
$T_1 \cup T_2 =$ 

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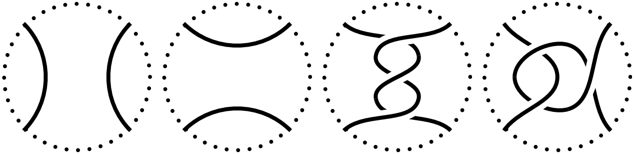

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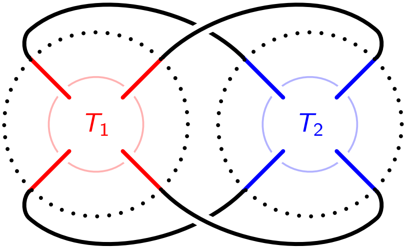
$T_1 \cup T_2 =$ 

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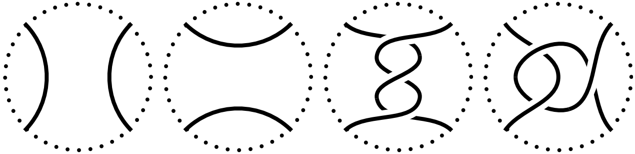

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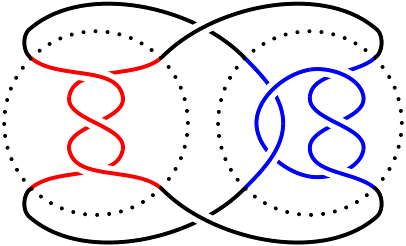

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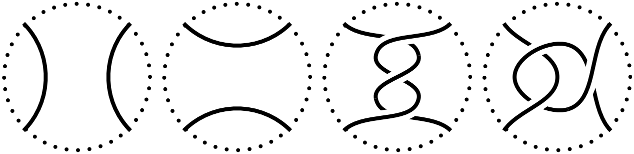

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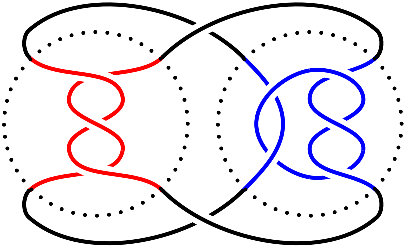
$Q_{-1/3} \cup T =$ 

Theorem [Kotelskiy-Watson-Z'20]

For any four-ended tangle T,

$$\mathrm{A}(T) := \{s \in \mathbb{Q}\mathrm{P}^1 \mid Q_{-s} \cup T \textit{ is an A-link}\}$$

is either $\varnothing$, a single point or an interval in $\mathbb{Q}\mathrm{P}^1$.

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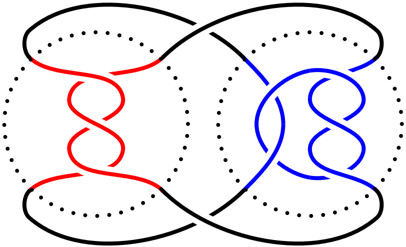
$Q_{-1/3} \cup T =$ 

Theorem [Kotelskiy-Watson-Z'20]

For any four-ended tangle T,

$$\mathrm{A}(T) := \{s \in \mathbb{Q}P^1 \mid Q_{-s} \cup T \text{ is an A-link}\}$$

is either $\varnothing$, a single point or an interval in $\mathbb{Q}P^1$. Moreover,

$$\Theta(T) := \{s \in \mathbb{Q}P^1 \mid Q_{-s} \cup T \text{ is thin}\}$$

is either $\varnothing$, a single point, two points or an interval in $\mathbb{Q}P^1$.

A-link fillings

Theorem [Rasmussen-Rasmussen'16]

For any 3-manifold M with torus boundary,

$$\mathcal{L}(M) = \{s \in \mathbb{Q}\mathrm{P}^1 \mid M(s) \text{ is an L-space}\}$$

is either $\varnothing$, a single point, a closed interval or $\mathbb{Q}\mathrm{P}^1 \setminus \{\lambda\}$.

Theorem [Kotelskiy-Watson-Z'20]

For any four-ended tangle T,

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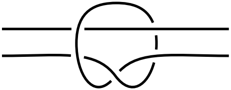
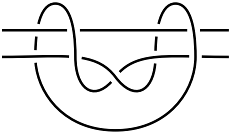
is either $\varnothing$, a single point, two points or an interval in $\mathbb{Q}\mathrm{P}^1$.

A-link/thin filling spaces: examples

tangle	$\Theta(T)$
	$\mathbb{Q}P^1 \setminus \{0\}$
	$[\infty, 2)$
	$\{\infty\}$
$T \# K$ for K thick	$\varnothing$

Note: This is for both $\widehat{\mathrm{HFK}}$ and $\widetilde{\mathrm{Kh}}$.

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A-link gluing theorem

Theorem [Hanselman-Rasmussen-Watson'16]

Let M and M' be two 3-manifolds with torus boundary and $h\colon \partial M \xrightarrow{\cong} \partial M'$, which are boundary incompressible. Then

$$h(\mathring{\mathcal{L}}(M)) \cup \mathring{\mathcal{L}}(M') = \mathbb{Q}\mathrm{P}^1 \Leftrightarrow M \cup_h M' \text{ is an L-space.}$$

Theorem [Kotelskiy-Watson-Z'20]

For any four-ended tangles T_1 and T_2,

$$\Big(\mathrm{m}(\mathring{\Theta}(T_1)) \cup \mathring{\Theta}(T_2) = \mathbb{Q}\mathrm{P}^1 \Big) \Rightarrow T_1 \cup T_2 \text{ is thin.}$$

Similarly,

$$\Big(\mathrm{m}(\mathring{\mathrm{A}}(T_1)) \cup \mathring{\mathrm{A}}(T_2) = \mathbb{Q}\mathrm{P}^1 \Big) \Rightarrow T_1 \cup T_2 \text{ is an A-link}$$

For the second implication and $\mathrm{H}_* = \widetilde{\mathrm{Kh}}$, we have to assume that a certain condition is satisfied for $\mathrm{H}_*(T_1)$ and $\mathrm{H}_*(T_2)$.

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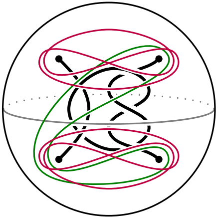
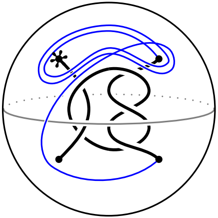
Immersed curve invariants for four-ended tangles



$\mathrm{HFT}(T)$



$\widetilde{\mathrm{BN}}(T)$

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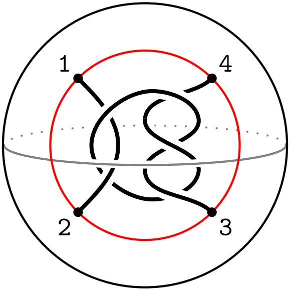
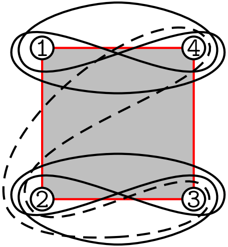
The immersed curve invariant HFT [Z'17]

$$\frac{\left\{\begin{array}{c}\text{4-ended}\\ \text{tangles } T \subset D^3\end{array}\right\}}{\text{isotopy}} \longrightarrow \frac{\left\{\begin{array}{c}\text{immersed curves* on}\\ S^2 \smallsetminus 4 = \partial D^3 \smallsetminus \partial T\end{array}\right\}}{\text{homotopy}}$$

$$T \longmapsto \mathrm{HFT}(T)$$



$\longmapsto$



*) plus local systems $X \in \mathrm{GL}_n(\mathbb{F}_2)$

Thin links and Conway spheres

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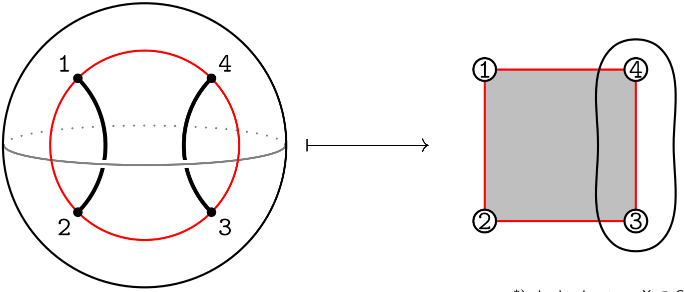

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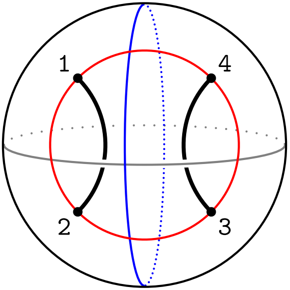

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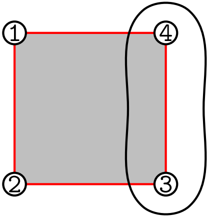

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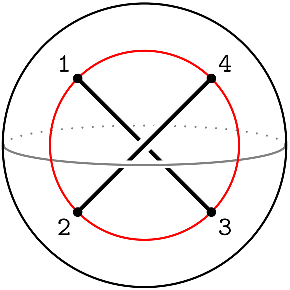

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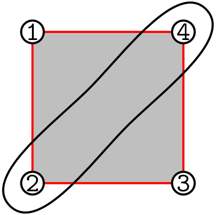

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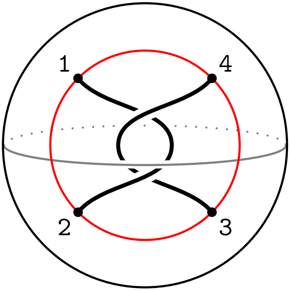

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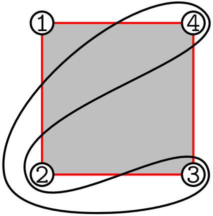

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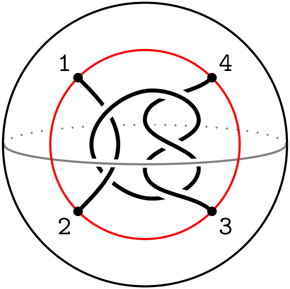

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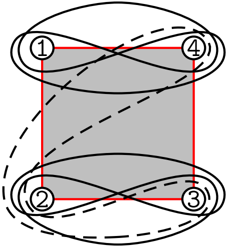

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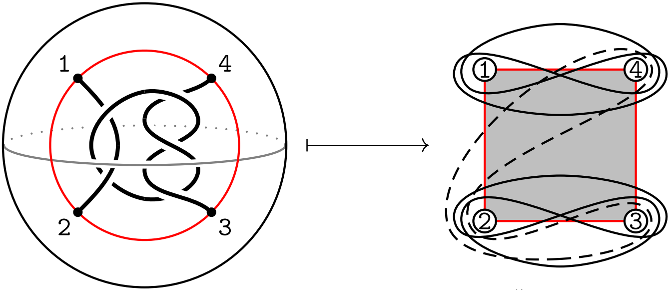

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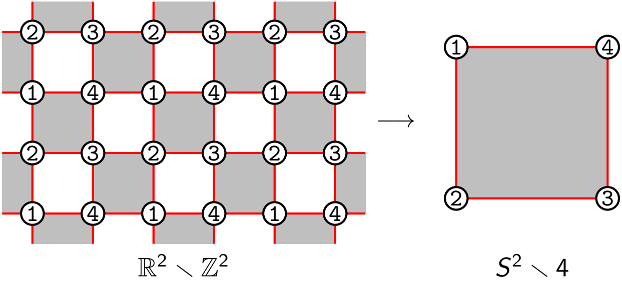
Theorem (gluing)

$\widehat{\text{HFL}}(T_1 \cup T_2) =$ *Lagrangian Floer homology of* $\text{m}\,\text{HFT}(T_1)$ *and* $\text{HFT}(T_2)$

A covering space for HFT



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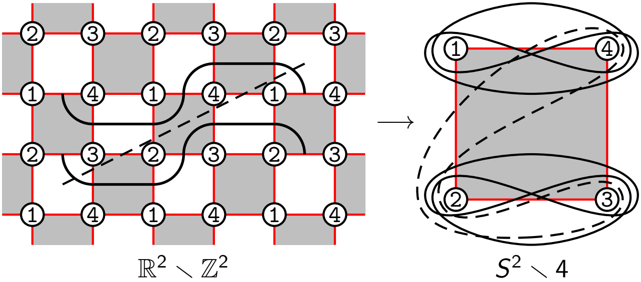

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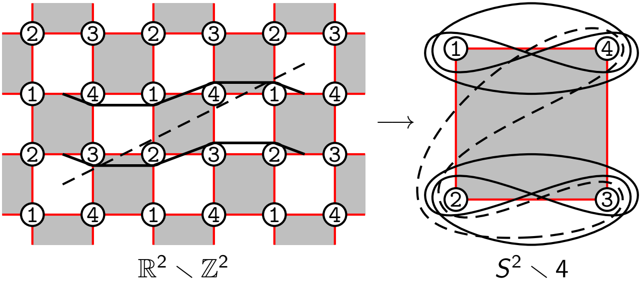

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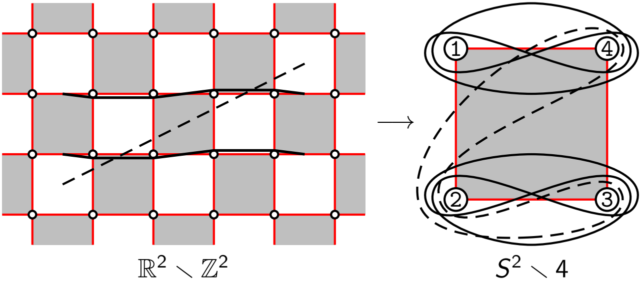

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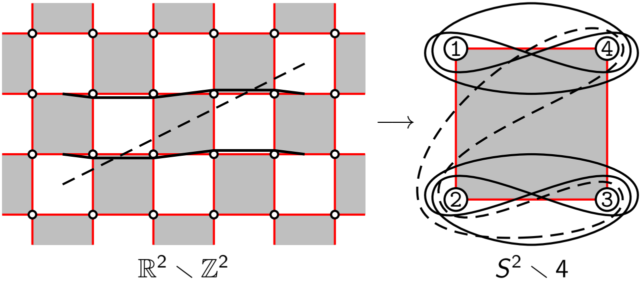

Lemma [Z'19]

All components of HFT(T) *lift to linear curves.*

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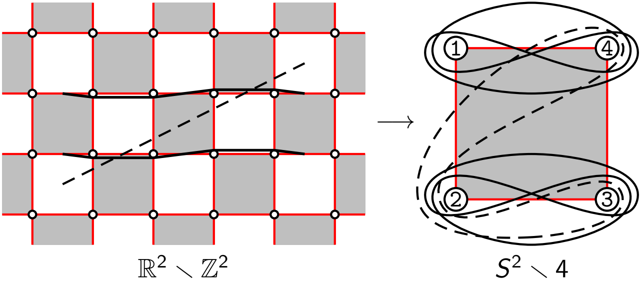

Lemma [Z'19]

All components of HFT(T) *lift to linear curves. Moreover, there are only two types of linear curves: rational and special.*

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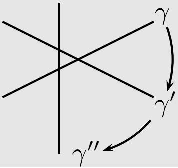
δ-grading on HFT

Lemma [Kotelskiy-Watson-Z'20]

The Lagrangian Floer homology $\mathrm{HF}(\gamma,\gamma')$ *of any two linear curves* γ *and* γ' *of distinct slopes* $s(\gamma)$, $s(\gamma')$ *is thin.*
Let $\delta(\gamma,\gamma') \coloneqq \delta$*-grading of* $\mathrm{HF}(\gamma,\gamma')$*. Then*

- $\delta(\gamma',\gamma) = 1 - \delta(\gamma,\gamma')$ *and*
- $\delta(\gamma,\gamma') + \delta(\gamma',\gamma'') = \delta(\gamma,\gamma'')$
  if $s(\gamma) > s(\gamma') > s(\gamma'') > s(\gamma)$



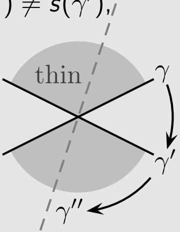
Example

Let $\Gamma = \gamma \amalg \gamma'$. Then for any γ'' with $s(\gamma) \neq s(\gamma'') \neq s(\gamma')$,

$\mathrm{HF}(\Gamma,\gamma'')$ is thin

$\Leftrightarrow \ \delta(\gamma,\gamma'') = \delta(\gamma',\gamma'')$

$$\Leftrightarrow \ \begin{cases} \delta(\gamma,\gamma') = 0 & s(\gamma) > s(\gamma') > s(\gamma'') > s(\gamma) \\ \delta(\gamma',\gamma) = 0 & s(\gamma) < s(\gamma') < s(\gamma'') < s(\gamma) \end{cases}$$



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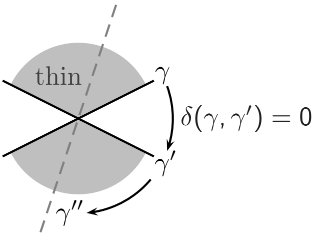

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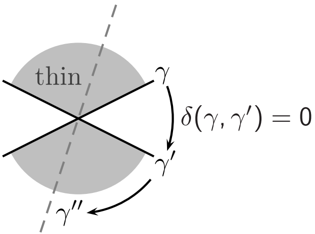

Theorem [Kotelskiy-Watson-Z'20]

For any finite collection Γ *of linear curves,*

$$\Theta(\Gamma) = \{s \in \mathbb{Q}\mathrm{P}^1 \mid \mathrm{HF}(\Gamma,\gamma) \textit{ thin for any linear } \gamma \textit{ of slope } s\}$$

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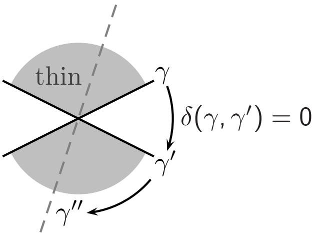

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is an open (possibly empty) interval in $\mathbb{Q}\mathrm{P}^1$.

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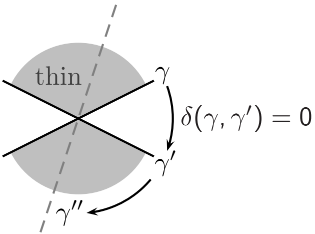

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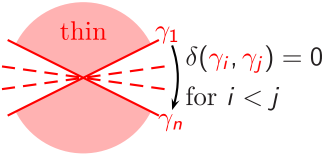

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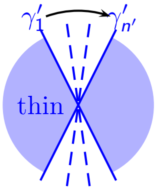

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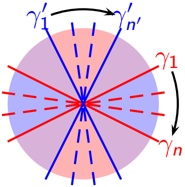

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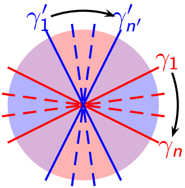

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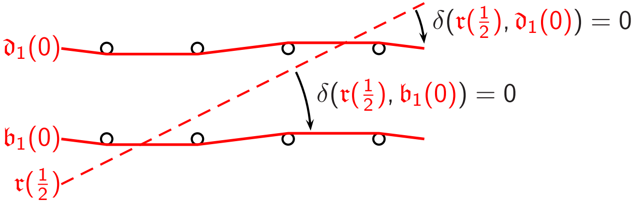

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$\Leftarrow ?\quad \Theta(\mathrm{HFT}($$)) = (\frac{1}{2}, 0)$

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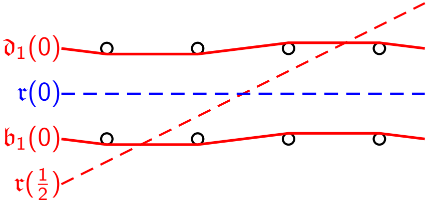

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$\Leftarrow ?$ Θ(HFT()) $= (\frac{1}{2}, 0)$, but HF(HFT(), $\mathfrak{r}(0)$) is thin.

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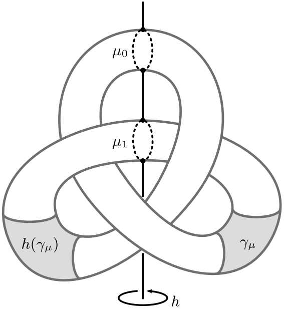
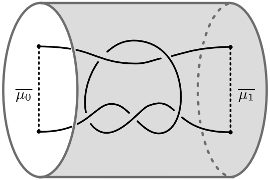
L-space knots and A-link tangles

figure taken from [Liam Watson. *Khovanov homology and the symmetry group of a knot*, Advances in Mathematics, Volume 313 (2017)]



$\mathcal{L}(RHT) = [1, \infty]$



$\Theta(T_{2,-3}) = (4, \infty]$

Conjecture [Liam Watson]

A strongly invertible knot in S^3 is an L-space knot if and only if the corresponding quotient is an A-link tangle.

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