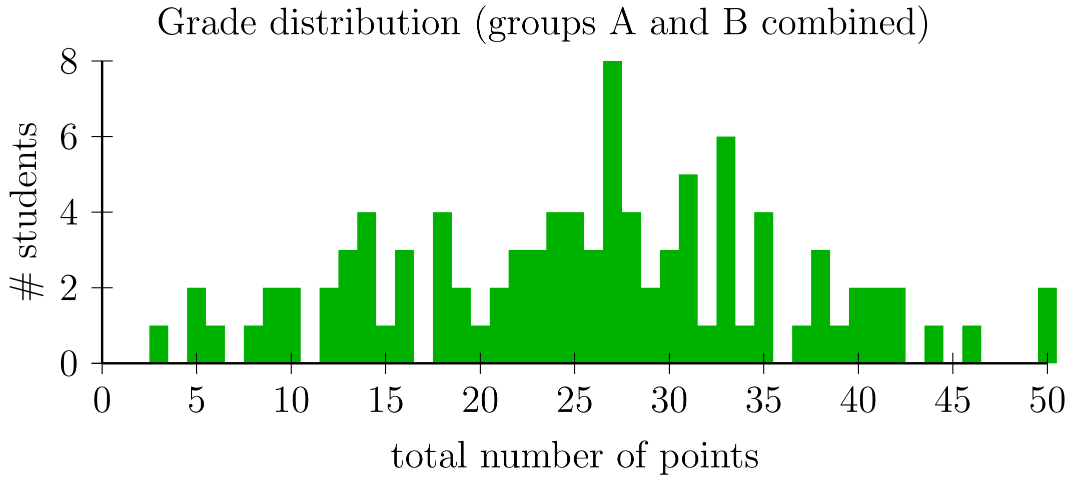
mathematics 100 section 102 mid-term exam

solutions and results



detailed results (group A)

question	1	2	3	4	5	6	7	8
average grade	90%	51%	71%	37%	40%	27%	42%	47%

detailed results (group B)

question	1	2	3	4	5	6	7	8
average grade	52%	39%	29%	36%	35%	72%	62%	82%

Note: The questions for groups A and B are very similar, but they are in opposite order. The maximal discrepancy between the results of the two groups is 11 percentage points in question 2/7 (group A/B).

General Comments

- Failure is not final. If you did not do well in this mid-term exam, there is still the final exam. **If you do significantly better in the final exam, the midterm will count less that the announced 25% towards the final grade.** (The exact change of the percentages is at the IIC's discretion.)
- A surprisingly large number of you still struggle with basic algebraic manipulation. Note the difference between
$$(\cos x)^{x^2} \quad \text{and} \quad ((\cos x)^x)^2$$
or
$$a + b \cdot x \quad \text{and} \quad (a + b) \cdot x$$
Often, the problem is just sloppiness/carelessness in notation. But I suspect that for some of you the problem is more severe than that. For example, in the past two days, I have seen things like
$$\frac{\cos(x^{19})}{x^3} = \cos(x^{16}).$$
Luckily, this was quite rare. But still, far too many people only got half the marks in question 7 (group A)/question 2 (group B) because they were unable to apply binomial identities and then simplify a simple fraction.
- Many people wrote in question 6 (group A)/question 3 (group B) that if a function is continuous, then it is also differentiable. **Think about this statement!**
- Many people got full marks or almost full marks on question 1 (group A)/question 8 (group B) and question 3 (group A)/question 6 (group B). **Well done!**
- I uploaded your grades to Canvas on Friday; however, they will not be released until the midterm exams of the other sections have also been marked and uploaded.

Comments about solutions and graded exams

- The solutions below come with short marking schemes that will hopefully explain some of the comments on your marked exams, when I return them on Tuesday.
- If you find mistakes in these solutions, do let me know!

solutions group A

5 marks

1. Find the derivative of

$$f(x) = \sqrt{42 - 2x}$$

using the definition of derivative. **No credit will be given for using differentiation rules, but you may use differentiation rules to check your answer.**

solution **[2 definition + 2 binomial + 1 accuracy]**

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{42-2x-2h} - \sqrt{42-2x}}{h} = \lim_{h\to 0} \frac{-2h}{h(\sqrt{42-2x-2h} + \sqrt{42-2x})} = -\frac{1}{\sqrt{42-2x}}$$

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7 marks

2. Let $f(x) = (\cos x)^{x^2}$. Compute $f'(x)$.

solution **[2 chain rule + 2 product rule + 3 accuracy]**

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2\log(\cos x))}\right) = (\cos x)^{x^2} \cdot \left(2x\log(\cos x) - x^2 \cdot \frac{\sin(x)}{\cos(x)}\right)$$

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8 marks

3. Find an equation of the straight line tangent to the curve

$$x^4 - x^2y + y^4 = 1$$

at the point $(x, y) = (-1, 1)$.

solution **[3+1+2 (per step) +1 accuracy + 1 complete answer]**

Differentiate:

$$4x^3 - 2xy - x^2y' + 4y^3y' = 0$$

Substitute $(x, y) = (-1, 1)$:

$$-4 + 2 - y' + 4y' = 0$$

Solve for y' and find equation:

$$y' = \tfrac{2}{3}, \text{ so } y = \tfrac{2}{3}(x+1) + 1$$

$\boxed{\text{3 marks}}$ 4. Compute

$$\arcsin\left(\sin\left(\frac{23\pi}{11}\right)\right).$$

solution **[2 using correct definition + 1 accuracy]**

$$\frac{23\pi}{11} = 2\pi + \frac{\pi}{11} \text{ and } -\frac{\pi}{2} \leq \frac{\pi}{11} \leq \frac{\pi}{2},$$

The range of arcsin is the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so the answer is $\frac{\pi}{11}$.

$\boxed{\text{6 marks}}$ 5. Evaluate

$$\lim_{a\to\infty} \frac{\sin(a^{19} + 19\log(a))}{a^2 - 38}$$

or determine that this limit does not exist.

solution **[4 successful application of theorem + 2 accuracy]**

$$-\frac{1}{a^2 - 38} \leq \frac{\sin(a^{19} + 19\log(a))}{a^2 - 38} \leq \frac{1}{a^2 - 38}$$

so by the squeezing theorem,

$$0 = \lim_{a\to\infty} -\frac{1}{a^2 - 38} \leq \lim_{a\to\infty} \frac{\sin(a^{19} + 19\log(a))}{a^2 - 38} \leq \lim_{a\to\infty} \frac{1}{a^2 - 38} = 0.$$

So

$$\lim_{a\to\infty} \frac{\sin(a^{19} + 19\log(a))}{a^2 - 38} = 0.$$

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8 marks 6. Find all real values of a for which the function

$$f(x) = \begin{cases} a \cdot x^2 & x \leq \frac{2}{\pi} \\ x \cdot \sin(\frac{1}{x}) & x > \frac{2}{\pi} \end{cases}$$

is continuous at $x = \frac{2}{\pi}$. Furthermore, determine if there are any values of a for which $f(x)$ is differentiable at $x = \frac{2}{\pi}$, and if so, find all such values a. **Justify your answers.**

solution[2 cont. + 2 diff. ⇒ cont. + 2 diff. + 1 accuracy + 1 complete answer]

$$\lim_{x \to \frac{2}{\pi}+} f(x) = \lim_{x \to \frac{2}{\pi}+} x \cdot \sin(\tfrac{1}{x}) = \tfrac{2}{\pi} \cdot \sin(\tfrac{\pi}{2}) = \tfrac{2}{\pi} \overset{!}{=} f(\tfrac{2}{\pi}) = a \cdot (\tfrac{2}{\pi})^2,$$

so $f(x)$ is continuous at $x = \frac{2}{\pi}$ iff $a = \frac{\pi}{2}$. Furthermore, f can only be differentiable if f is also continuous; so it only remains to see if f is differentiable at $x = \frac{2}{\pi}$ for $a = \frac{\pi}{2}$. If it were, it would agree with both

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{\pi}{2} \cdot x^2\right)\Big|_{x=\frac{2}{\pi}} = \pi \cdot x\big|_{x=\frac{2}{\pi}} = 2$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \cdot \sin(\tfrac{1}{x})\right)\Big|_{x=\frac{2}{\pi}} = \left(\sin(\tfrac{1}{x}) - \frac{x}{x^2}\cos(\tfrac{1}{x})\right)\Big|_{x=\frac{2}{\pi}} = \sin(\tfrac{\pi}{2}) - \tfrac{\pi}{2} \cdot \underbrace{\cos(\tfrac{\pi}{2})}_{0} = 1.$$

So $f(x)$ is not differentiable at $x = \frac{2}{\pi}$ for any value of a.

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8 marks 7. Find the derivative of the function

$$f(x) = \frac{e \cdot (e^x - e^{-x})}{e^x + e^{-x}}$$

and simplify the result as much as possible.

solution **[4 quotient rule + 4 accuracy]**

$$f'(x) = e \cdot \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4e}{(e^x + e^{-x})^2}$$

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5 marks 8. Evaluate

$$\lim_{x \to 2} \frac{x^3 - x^2 + x - 6}{x^2 + x - 6},$$

without using l'Hôpital's rule. (Do not worry if you do not know this rule, it will be discussed in the second half of term.)

solution **[2+1 (factorisations) + 1 limit + 1 complete answer]**

$$x^3 - x^2 + x - 6 = x^3 - 2x^2 + x^2 - 2x + 3x - 6 = (x-2)(x^2 + x + 3)$$

and

$$x^2 + x - 6 = (x-2)(x+3)$$

so

$$\lim_{x \to 2} \frac{x^3 - x^2 + x - 6}{x^2 + x - 6} = \lim_{x \to 2} \frac{x^2 + x + 3}{x + 3} = \frac{9}{5} = 1.8$$

solutions group B

$\boxed{\text{5 marks}}$ 1. Evaluate

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - x - 6}{x^2 - x - 6},$$

without using l'Hôpital's rule. (Do not worry if you do not know this rule, it will be discussed in the second half of term.)

solution **[2+1 (factorisations) + 1 limit + 1 complete answer]**

$$x^3 - 2x^2 - x - 6 = x^3 - 3x^2 + x^2 - 3x + 2x - 6 = (x-3)(x^2+x+2)$$

and

$$x^2 - x - 6 = (x+2)(x-3)$$

so

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - x - 6}{x^2 - x - 6} = \lim_{x\to 3} \frac{x^2 + x + 2}{x+2} = \frac{14}{5} = 2.8$$

$\boxed{\text{8 marks}}$ 2. Find the derivative of the function

$$f(x) = \frac{(e^x + e^{-x})}{e \cdot (e^x - e^{-x})}$$

and simplify the result as much as possible.

solution **[4 quotient rule + 4 accuracy]**

$$f'(x) = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{e \cdot (e^x - e^{-x})^2} = -\frac{4}{e(e^x - e^{-x})^2}$$

$\boxed{\text{8 marks}}$ 3. Find all real values of a for which the function

$$f(x) = \begin{cases} a \cdot x^2 & x \leq \frac{1}{\pi} \\ x \cdot \cos(\frac{1}{x}) & x > \frac{1}{\pi} \end{cases}$$

is continuous at $x = \frac{1}{\pi}$. Furthermore, determine if there are any values of a for which $f(x)$ is differentiable at $x = \frac{1}{\pi}$, and if so, find all such values a. **Justify your answers.**

solution[2 cont. + 2 diff. ⇒ cont. + 2 diff. + 1 accuracy + 1 complete answer]

$$\lim_{x \to \frac{1}{\pi}+} f(x) = \lim_{x \to \frac{1}{\pi}+} x \cdot \cos(\tfrac{1}{x}) = \tfrac{1}{\pi} \cdot \cos(\pi) = -\tfrac{1}{\pi} \overset{!}{=} f(\tfrac{1}{\pi}) = a \cdot \tfrac{1}{\pi^2},$$

so $f(x)$ is continuous at $x = \frac{1}{\pi}$ iff $a = -\pi$. Furthermore, f can only be differentiable if f is also continuous; so it only remains to see if f is differentiable at $x = \frac{1}{\pi}$ for $a = -\pi$. If it were, it would agree with both

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\pi \cdot x^2\right)\bigg|_{x=\frac{1}{\pi}} = -2\pi \cdot x\big|_{x=\frac{1}{\pi}} = -2$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \cdot \cos(\tfrac{1}{x})\right)\bigg|_{x=\frac{1}{\pi}} = \left(\cos(\tfrac{1}{x}) + \frac{x}{x^2}\sin(\tfrac{1}{x})\right)\bigg|_{x=\frac{1}{\pi}} = \cos(\pi) + \pi \cdot \underbrace{\sin(\pi)}_{0} = -1.$$

So $f(x)$ is not differentiable at $x = \frac{1}{\pi}$ for any value of a.

$\boxed{\text{6 marks}}$ 4. Evaluate

$$\lim_{a \to \infty} \frac{\cos(a^{17} + 17\log(a))}{a^2 - 34}$$

or determine that this limit does not exist.

solution **[4 successful application of theorem + 2 accuracy]**

$$-\frac{1}{a^2 - 34} \leq \frac{\cos(a^{17} + 17\log(a))}{a^2 - 34} \leq \frac{1}{a^2 - 34}$$

so by the squeezing theorem,

$$0 = \lim_{a \to \infty} -\frac{1}{a^2 - 34} \leq \lim_{a \to \infty} \frac{\cos(a^{17} + 17\log(a))}{a^2 - 34} \leq \lim_{a \to \infty} \frac{1}{a^2 - 34} = 0.$$

So

$$\lim_{a \to \infty} \frac{\cos(a^{17} + 17\log(a))}{a^2 - 34} = 0.$$

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3 marks 5. Compute

$$\arccos\left(\cos\left(\frac{36\pi}{17}\right)\right).$$

solution **[2 using correct definition + 1 accuracy]**

$$\frac{36\pi}{17} = 2\pi + \frac{2\pi}{17} \text{ and } 0 \leq \frac{2\pi}{17} \leq \pi.$$

The range of arccos is the closed interval $[0, \pi]$, so the answer is $\frac{2\pi}{17}$.

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8 marks 6. Find an equation of the straight line tangent to the curve

$$x^4 + x^3y + y^4 = 1$$

at the point $(x, y) = (-1, 1)$.

solution **[3+1+2 (per step) +1 accuracy + 1 complete answer]**

Differentiate:

$$4x^3 + 3x^2y + x^3y' + 4y^3y' = 0$$

Substitute $(x, y) = (-1, 1)$:

$$-4 + 3 - y' + 4y' = 0$$

Solve for y' and find equation:

$$y' = \tfrac{1}{3}, \text{ so } y = \tfrac{1}{3}(x + 1) + 1$$

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7 marks 7. Let $f(x) = (\cos x)^{x^3}$. Compute $f'(x)$.

solution **[2 chain rule + 2 product rule + 3 accuracy]**

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3\log(\cos x)}\right) = (\cos x)^{x^3} \cdot \left(3x^2\log(\cos x) - x^3 \cdot \frac{\sin(x)}{\cos(x)}\right)$$

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5 marks 8. Find the derivative of

$$f(x) = \sqrt{36 - 4x}$$

using the definition of derivative. **No credit will be given for using differentiation rules, but you may use differentiation rules to check your answer.**

solution **[2 definition + 2 binomial + 1 accuracy]**

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{36 - 4x - 4h} - \sqrt{36 - 4x}}{h} = \lim_{h\to 0}\frac{-4h}{h(\sqrt{36 - 4x - 4h} + \sqrt{36 - 4x})} = -\frac{2}{\sqrt{36 - 4x}}$$